

Information, Persuasion, and Decisionmaking

Part 1: Decisionmaking under uncertainty

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Background

- Much of economic activity and strategic behavior centers around the **flow of information**

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Free Quick Auto Quote - Switch & Save - Discount Double Check - Find An Agent

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1 10 100 1000 Clear filter Order Delivery Order Pickup Make a Reservation

1. Hourly! Ray's
★★★★★ 1430 reviews
88 Southern, Chicken (Hot), American (Traditional)
Dine-in
727 N Broadway
Los Angeles, CA 90012
(213) 655-8386

This place is delicious! The staff is friendly, extremely welcoming - like you're a part of their family. I get here about an hour before opening and wait about another 45 minutes... read more

Background

- Much of economic activity and strategic behavior centers around the **flow of information**
- Traditional approaches: information in a game is fixed
- Reality: information can be actively designed/elicited/transferred
- **Recent, fundamental questions:**
 - How to reason about value of information?
 - How does information influence strategic behavior?
 - How to elicit valuable information from strategic sources?
 - How to design information structures to yield desired equilibrium?

Topics Covered in this Tutorial

- Signals as carriers of information, and their properties
- Single-agent decision problems and effect of information
- Bayesian games, equilibrium concepts, and effect of information
- Informational substitutes: definitions, applications, and algorithms
- Persuasion: models, algorithmic study, applications and generalizations
- Open problems and directions

Schedule of the Tutorial

8:30 am - 9:30 am

Part 1: Basics of decisionmaking under uncertainty

(short break)

9:40 am - 10:30 am

Part 2: Informational substitutes and complements

(10:30 am - 11:00 am coffee break)

11:00 am - 12:30 am

Part 3: Algorithmic persuasion

Outline of Part 1

- A. Model of information and signals
 - basic properties of signals

- B. Model of a single decisionmaker
 - basic properties of decision problems
 - how information impacts decisions
 - Blackwell ordering

- C. Bayesian games
 - equilibrium concepts

Part 1A:
Model of information and signals

Notation

- A set of actions agent chooses a
 - θ set of states of the world nature draws θ
 - $u(a, \theta)$ utility function
-
- p prior distribution on θ known to agent
 - Σ a signal (also refers to set of realizations) agent observes $\Sigma=\sigma$
 - $\varphi(\sigma, \theta)$ probability of signal σ given state θ
 - p_σ posterior distribution on θ given σ given by Bayes' rule
-

Basic properties of signals

Probability distributions, signals

- Agent starts with prior belief p in Δ_θ
- Agent observes signal σ from conditional distribution $\varphi(\sigma, \theta)$
- Agent updates to posterior belief p_σ using Bayes' rule

Fact 1

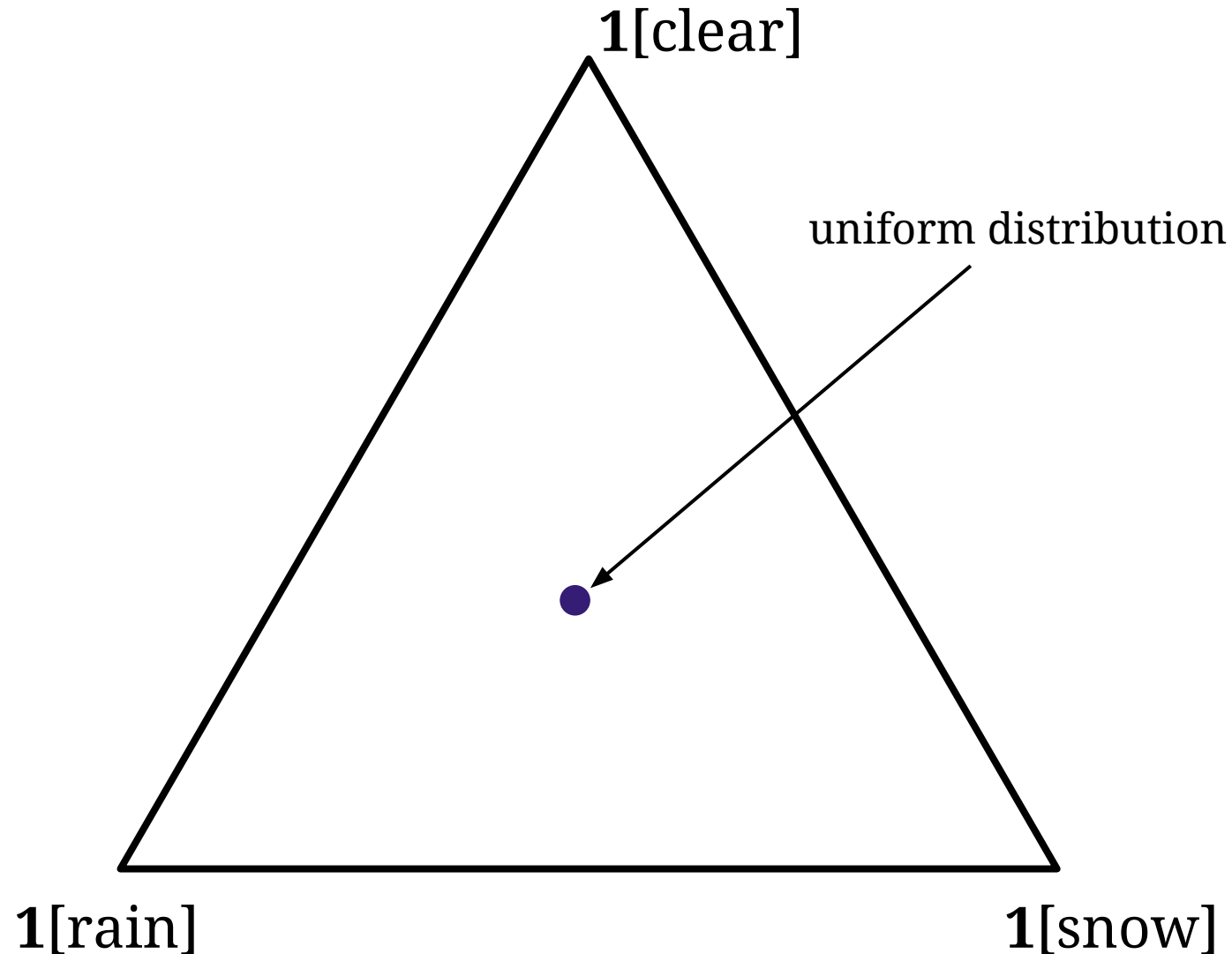
- (1) For all conditional distributions, $\mathbf{E}[p_\sigma] = p$.
(On average, the posterior equals the prior.)
(Your current belief is your expectation of your future belief.)
- (2) For any set of points $\{p_1, \dots, p_n\}$ such that p is in their convex hull, there exists a φ inducing this set of posterior beliefs.

Proof.

- (1) $\mathbf{E} \Pr[\theta | \sigma] = \sum_\sigma \Pr[\sigma] \Pr[\theta | \sigma] = \sum_\sigma \Pr[\theta, \sigma] = \Pr[\theta]$.
- (2) Write $p = \sum_\sigma \alpha_\sigma p_\sigma$ and let $\varphi(\sigma, \theta) = \alpha_\sigma p_\sigma(\theta) / p(\theta)$.

Example: a journey through Ithaca

Probability simplex on $\theta = \{\text{clear, rain, snow}\}$

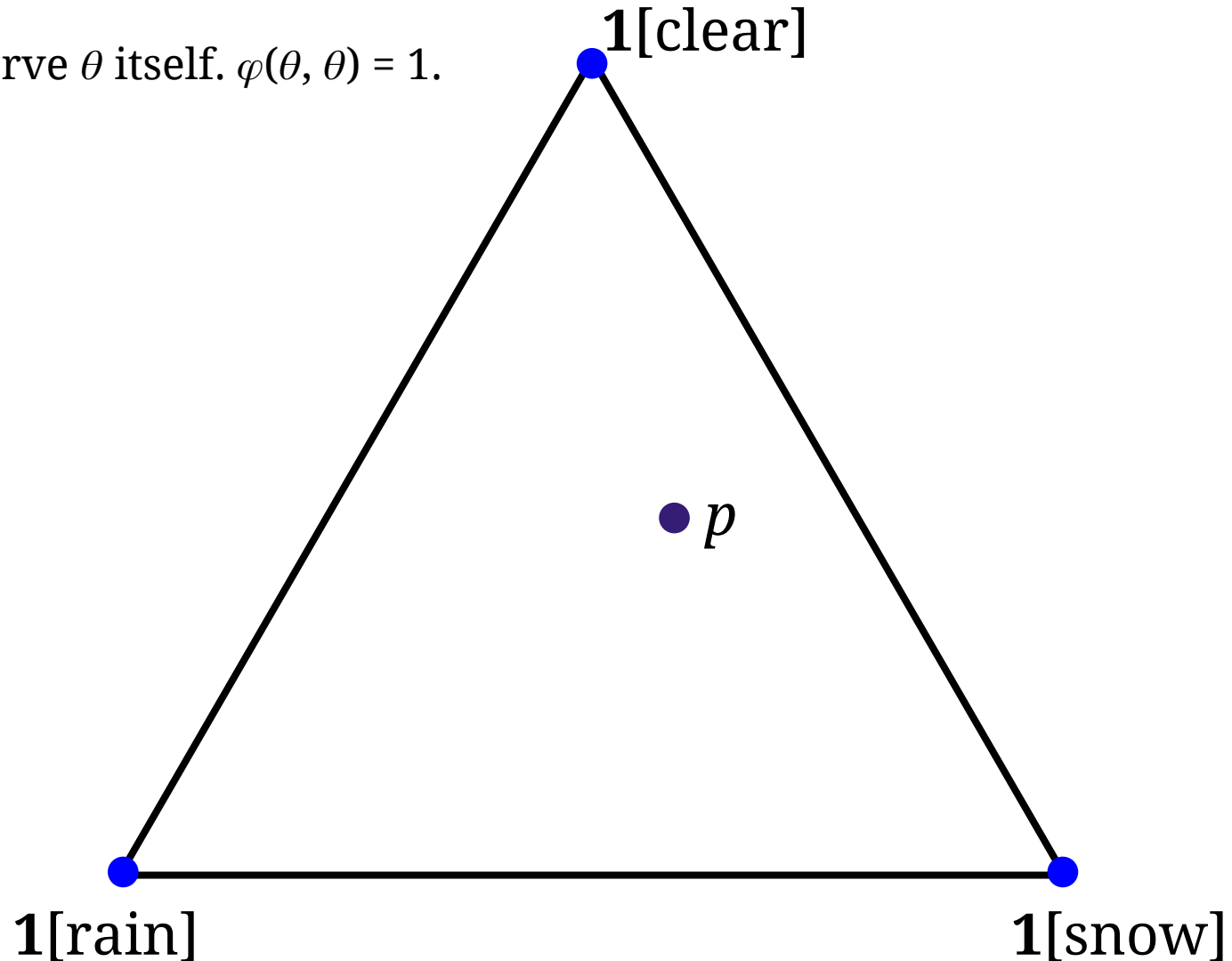


Example: a journey through Ithaca

Probability simplex on $\theta = \{\text{clear, rain, snow}\}$

Simplest signal: observe θ itself. $\varphi(\theta, \theta) = 1$.

- $p_{\text{rain}} = \mathbf{1}[\text{rain}]$ and so on.
- Fact 1.1: $p = \mathbf{E}[p_\theta]$.



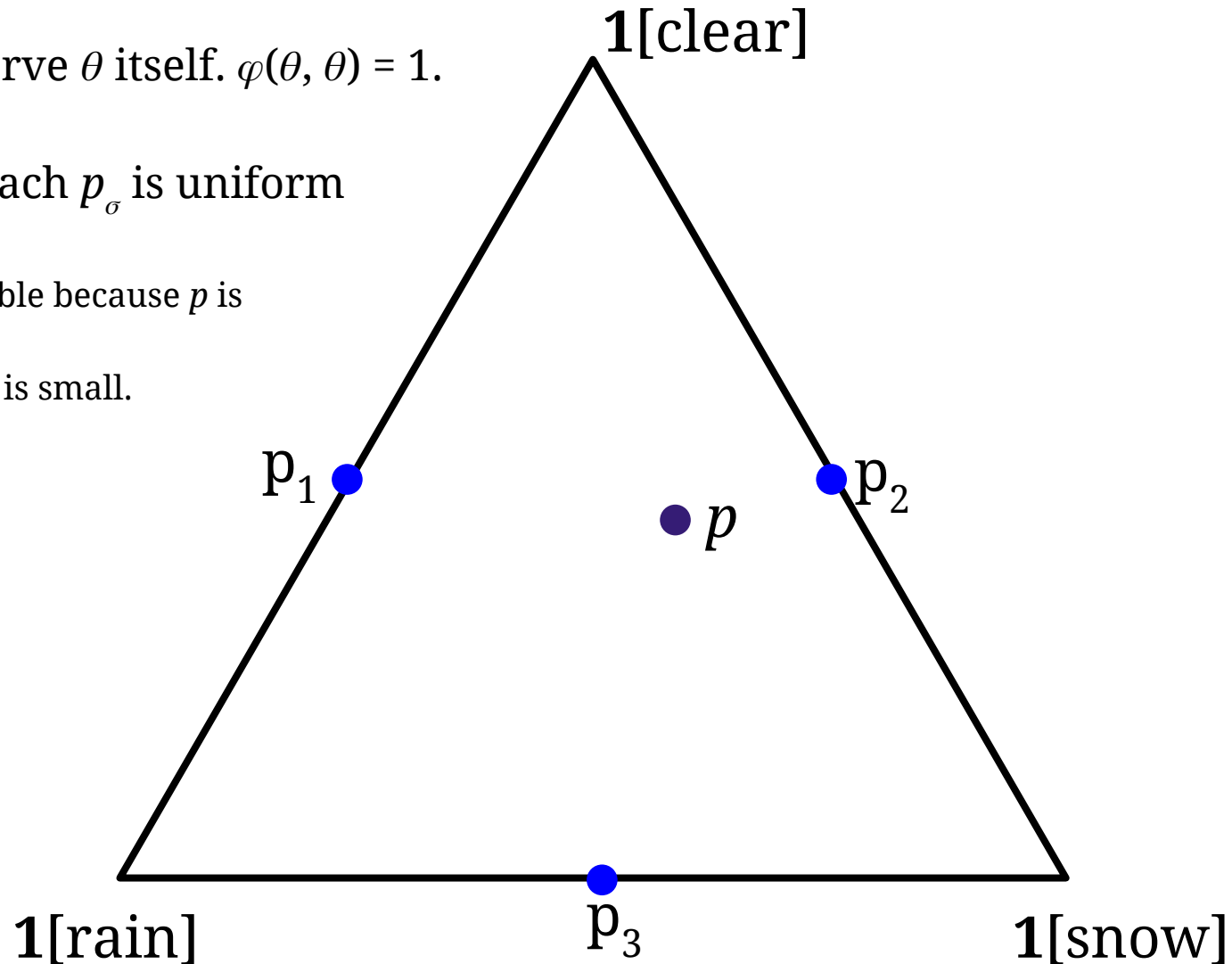
Example: a journey through Ithaca

Probability simplex on $\theta = \{\text{clear, rain, snow}\}$

Simplest signal: observe θ itself. $\varphi(\theta, \theta) = 1$.

An example where each p_σ is uniform on two of the states.

- Fact 1.2 says this is possible because p is in the convex hull.
- Fact 1.1 implies $\Pr[\sigma = 3]$ is small.



Example: a journey through Ithaca

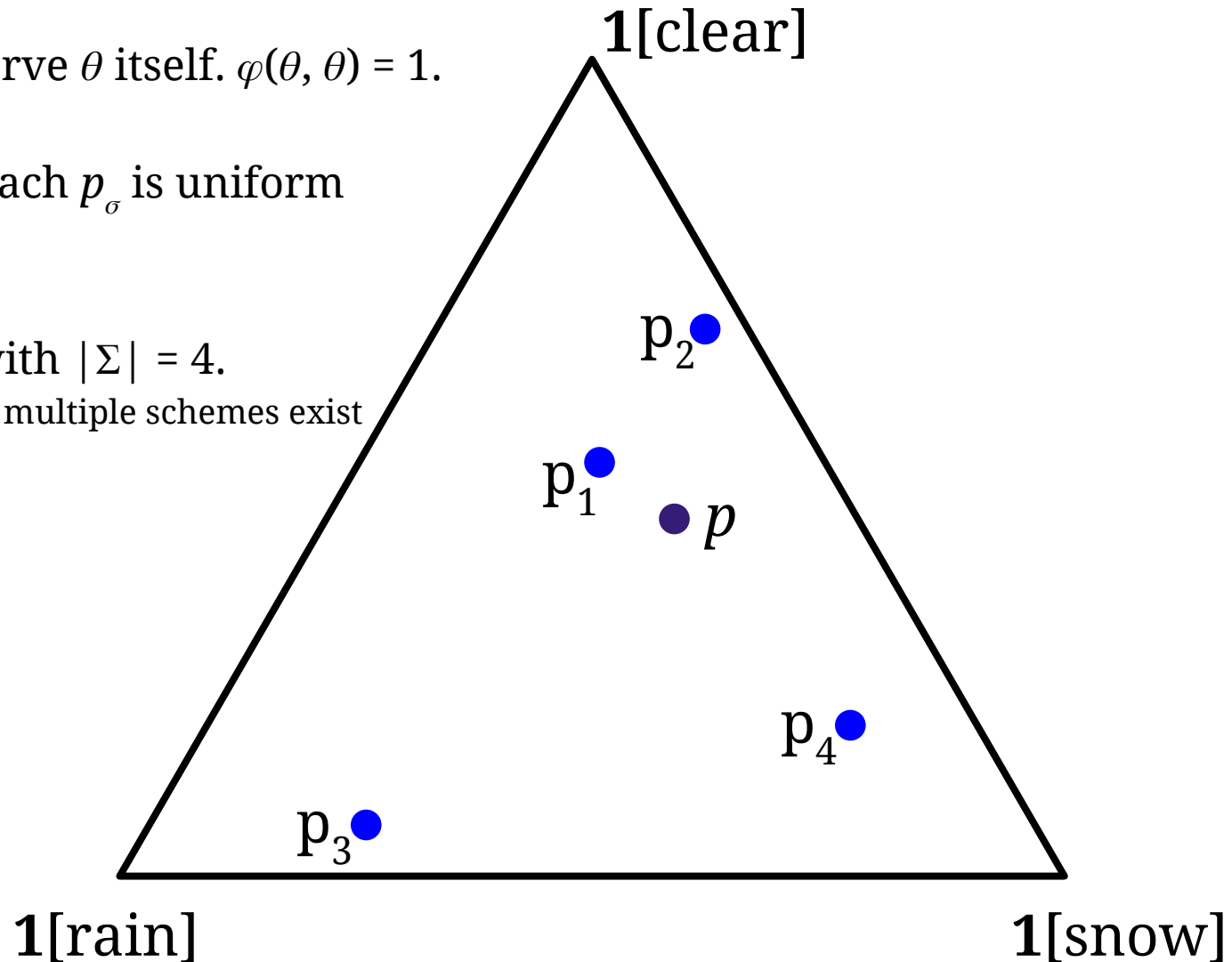
Probability simplex on $\theta = \{\text{clear}, \text{rain}, \text{snow}\}$

Simplest signal: observe θ itself. $\varphi(\theta, \theta) = 1$.

An example where each p_σ is uniform on two of the states.

A generic example with $|\Sigma| = 4$.

- System overdetermined; multiple schemes exist

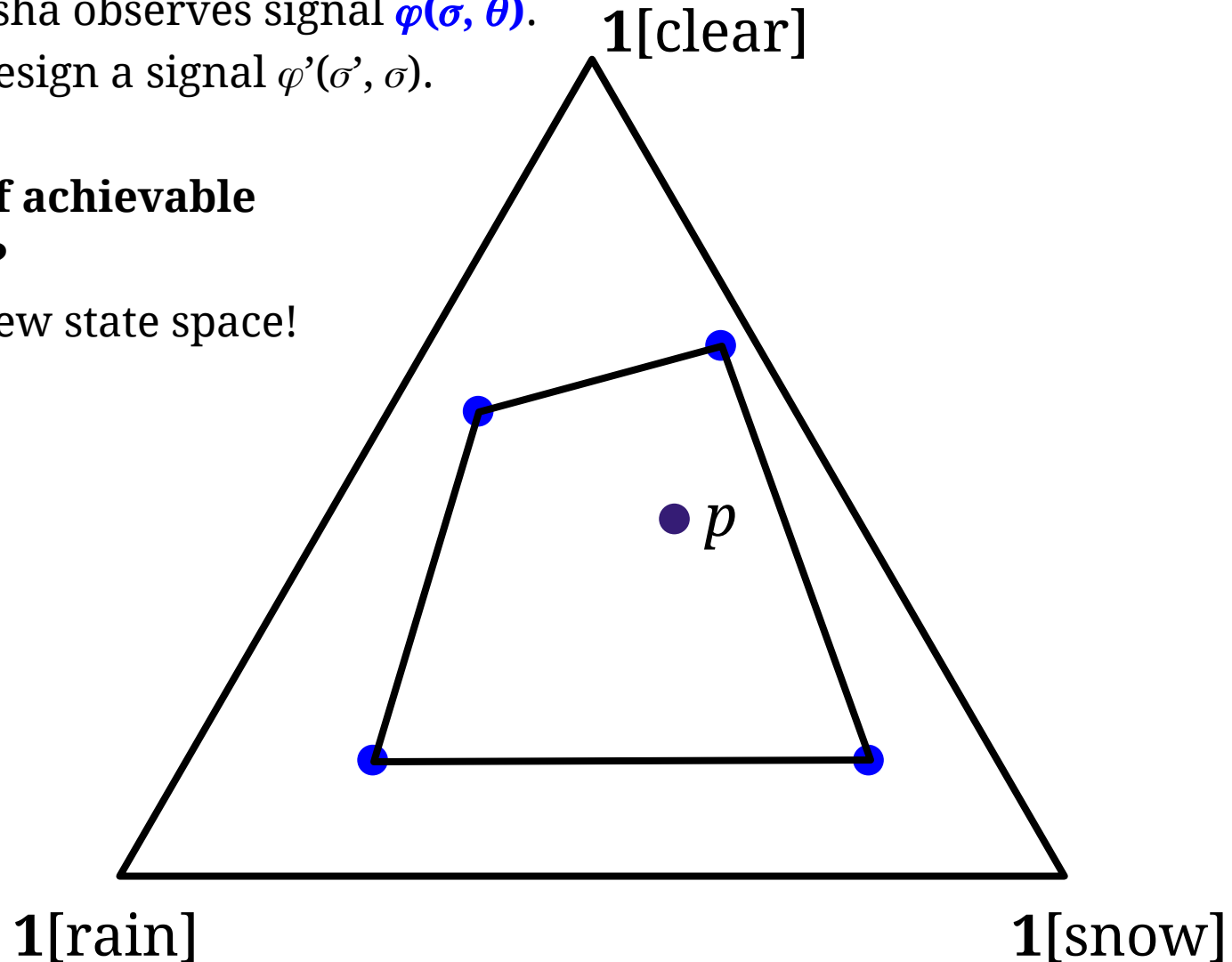


The Ithaca meteorologist

- The prior is p .
- Meteorologist Marsha observes signal $\varphi(\sigma, \theta)$.
- Marsha wants to design a signal $\varphi'(\sigma', \sigma)$.

What is the space of achievable signalling schemes?

- Think of Σ as the new state space!

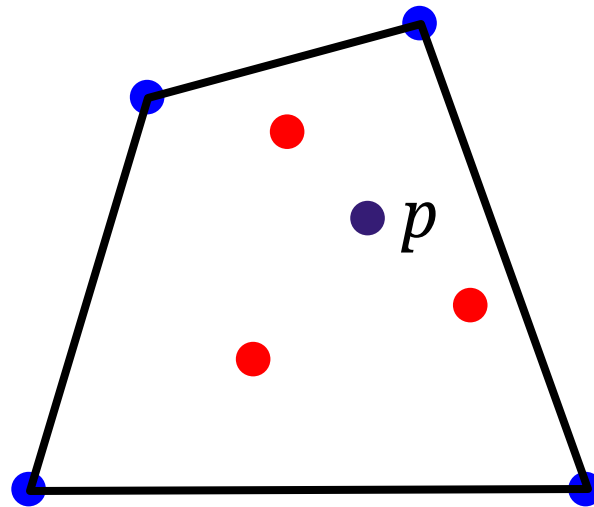


The Ithaca meteorologist

- The prior is p .
- Meteorologist Marsha observes signal $\varphi(\sigma, \theta)$.
- Marsha wants to design a signal $\varphi'(\sigma', \sigma)$.

What is the space of achievable signalling schemes?

- Think of Σ as the new state space!
(The convex hull of $\{p_\sigma\}$ is the new simplex.)
- Can easily characterize all schemes:
 - $\mathbf{E} p_\sigma = p$.
 - p must be in the convex hull of $\{p_\sigma\}$ which must be in the convex hull of $\{p\}$.
 - For each σ' , $\mathbf{E}[p_\sigma | \sigma'] = p_{\sigma'}$.



Therefore: It is without much loss to assume that a signaller observes the true state θ .

Part 1B: Decision problems

Notation

- A set of actions agent chooses a
 - Θ set of states of the world nature draws θ
 - $u(a, \theta)$ utility function
-
- p prior distribution on Θ known to agent
 - Σ a signal (also refers to set of realizations) agent observes $\Sigma=\sigma$
 - $\varphi(\sigma, \theta)$ probability of signal σ given state θ
 - p_σ posterior distribution on Θ given σ given by Bayes' rule
-
- $u(a ; q) = \mathbf{E}_{\theta \sim q} u(a, \theta)$ linear function of q
 - $G(q) = \max_a u(a ; q)$ convex function of q
 - $a^*(q) = \operatorname{argmax}_a u(a ; q)$ optimal action given q

Basics of decision problems

Decision problems and convex functions

- Agent must choose a based on belief q q may be prior or posterior
- Assume: chooses to maximize expected $_q$ utility
- Write $G(q) =$ “expected utility for optimal $_q$ action”

How to characterize all possible decision problems?

Decision problems and convex functions

- Agent must choose a based on belief q q may be prior or posterior
- Assume: chooses to maximize expected $_q$ utility
- Write $G(q) =$ “expected utility for optimal $_q$ action”

Fact 2

- (1) For every decision problem (A, θ, u) , $G(q) = \max_a u(a ; q)$ is **convex**.
- (2) Every convex $G : \Delta_\theta \rightarrow \mathbf{R}$ is the expected utility function for some decision problem (A, θ, u) .

Proof.

- (1) Each $u(a ; q)$ is a linear function of q ; a max of linear functions is convex.
- (2) We can write G as a maximum of linear functions of q .
Assign each linear function to an action a and write it as $u(a ; q)$.
Define $u(a, \theta) = u(a ; \mathbf{1}[\theta]) =$ expected utility for a under belief $\Pr[\theta] = \mathbf{1}$.

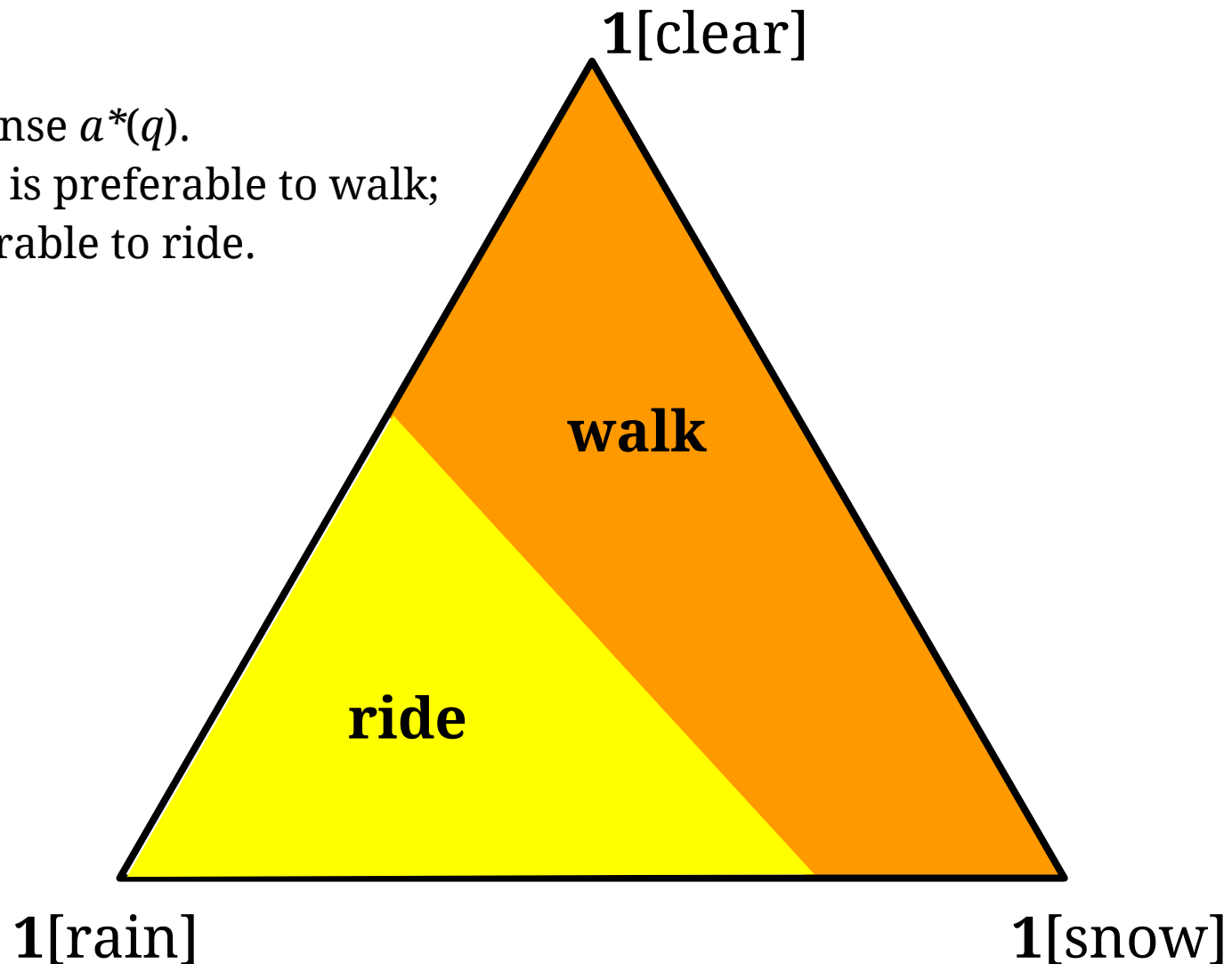
Example: a journey through Ithaca

Probability simplex on $\theta = \{\text{clear, rain, snow}\}$

$A = \{\text{walk, ride}\}$

Pictured: best-response $a^*(q)$.

For some beliefs q , it is preferable to walk;
for others, it is preferable to ride.



Example: a journey through Ithaca

Probability simplex on $\Theta = \{\text{clear, rain, snow}\}$

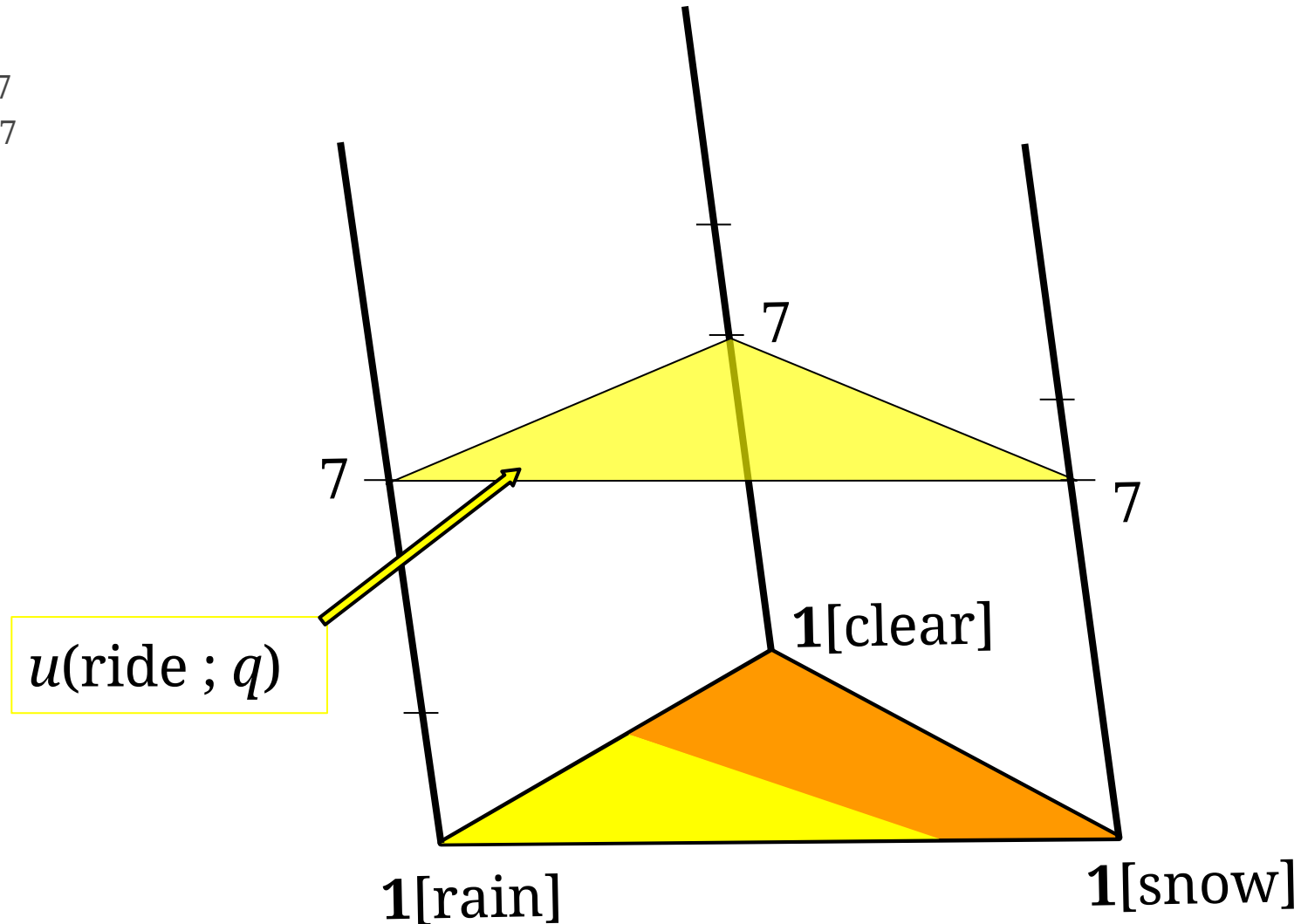
$A = \{\text{walk, ride}\}$

If we ride:

$$u(\text{ride, clear}) = 7$$

$$u(\text{ride, snow}) = 7$$

$$u(\text{ride, rain}) = 7$$



Example: a journey through Ithaca

Probability simplex on $\Theta = \{\text{clear, rain, snow}\}$

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If we ride:

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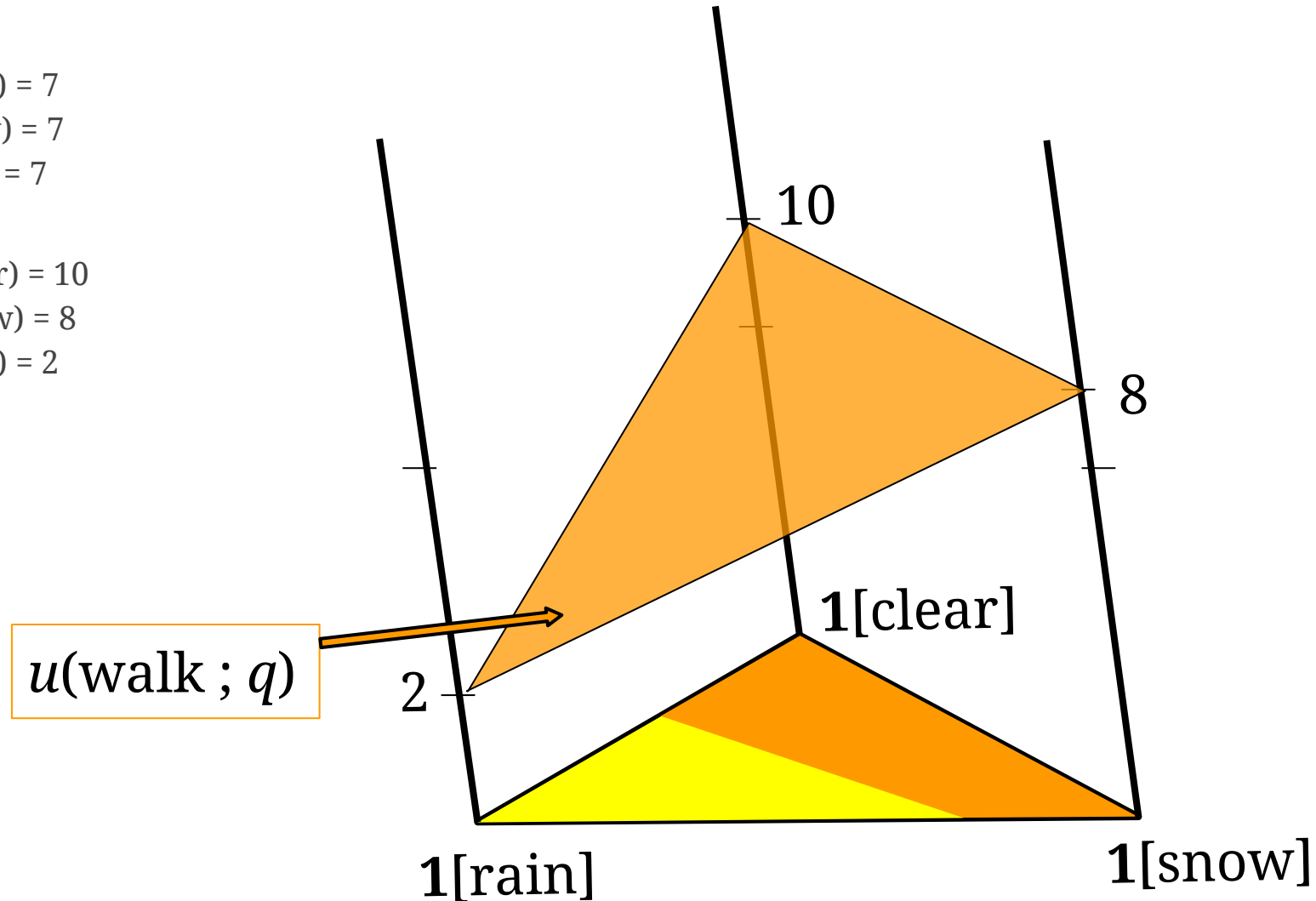
$$u(\text{ride, rain}) = 7$$

If we walk:

$$u(\text{walk, clear}) = 10$$

$$u(\text{walk, snow}) = 8$$

$$u(\text{walk, rain}) = 2$$



Example: a journey through Ithaca

Probability simplex on $\Theta = \{\text{clear, rain, snow}\}$

$A = \{\text{walk, ride}\}$

If we ride:

$$u(\text{ride, clear}) = 7$$

$$u(\text{ride, snow}) = 7$$

$$u(\text{ride, rain}) = 7$$

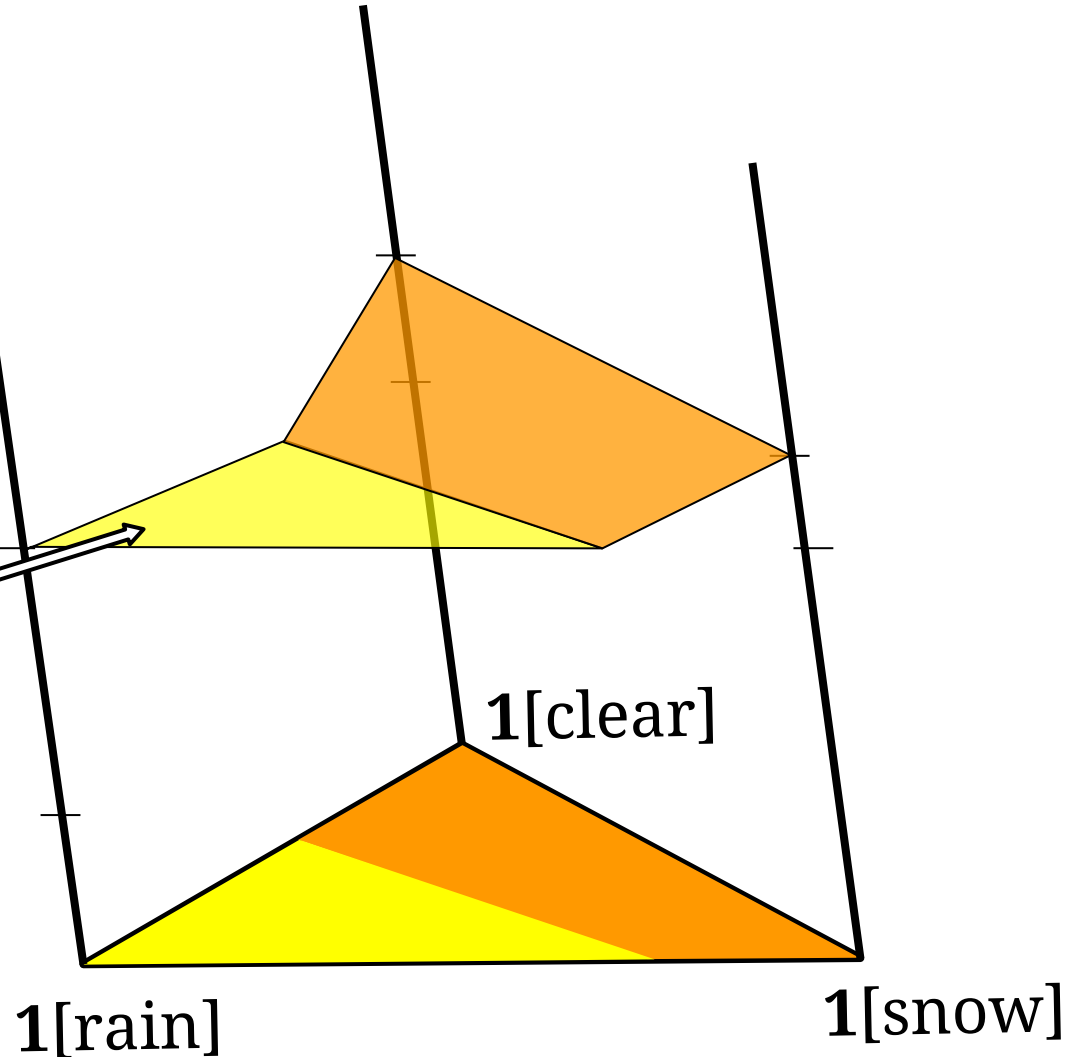
If we walk:

$$u(\text{walk, clear}) = 10$$

$$u(\text{walk, snow}) = 8$$

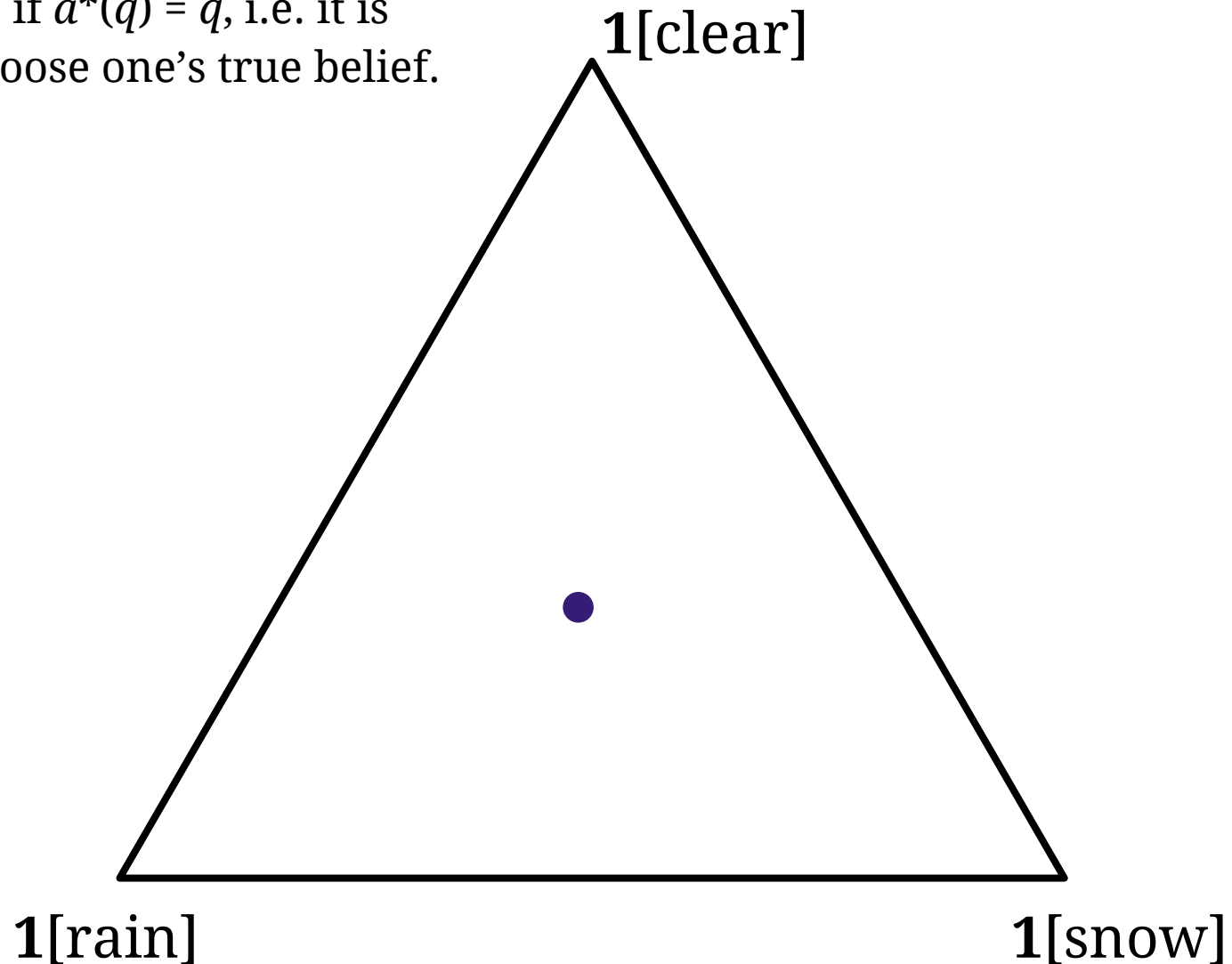
$$u(\text{walk, rain}) = 2$$

$G(q)$ = expected utility
for acting optimally
under belief q



Example: proper scoring rules

A decision problem $S(a, \theta)$ with $A = \Delta_\theta$ is a **proper scoring rule** if $a^*(q) = q$, i.e. it is always optimal to choose one's true belief.



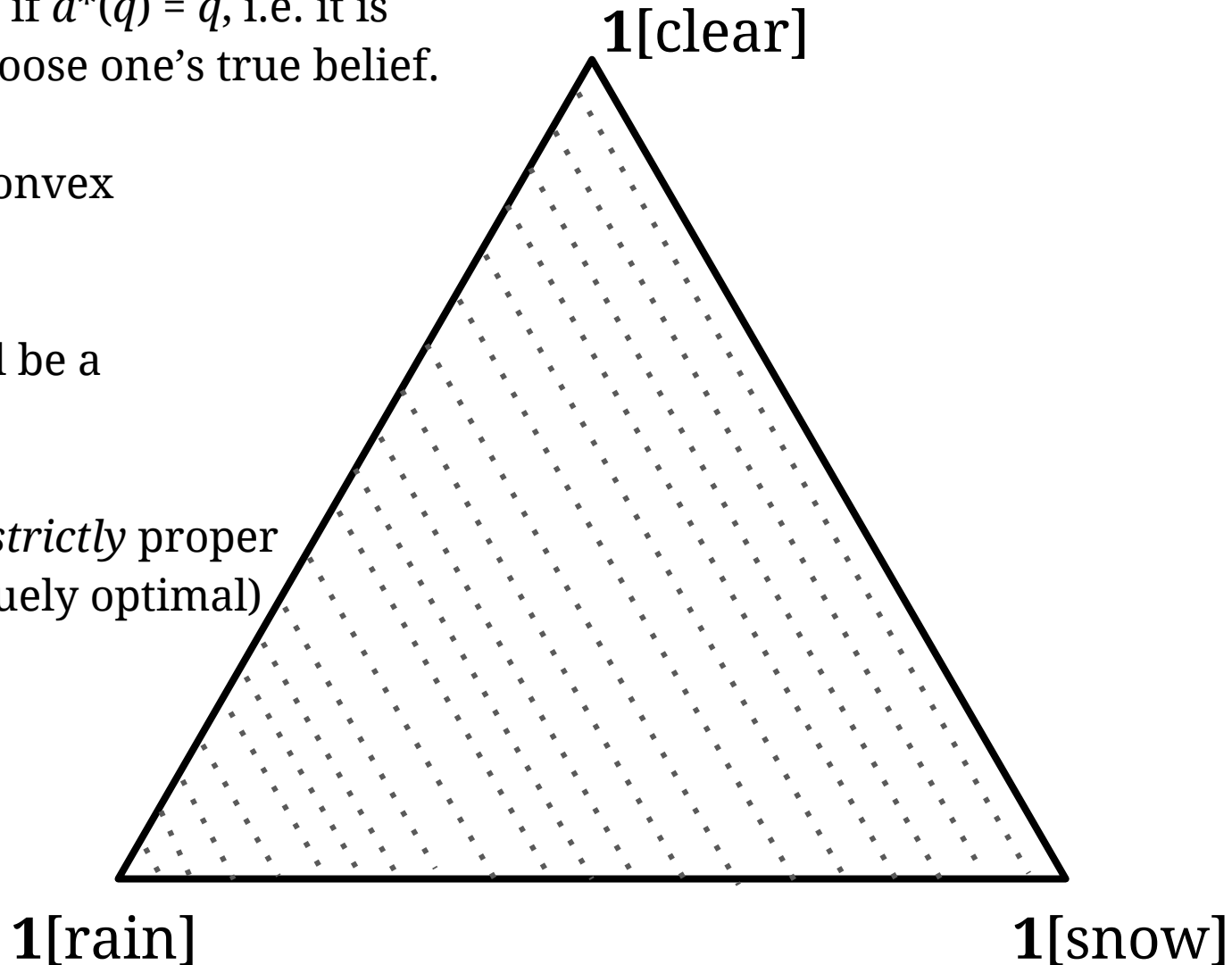
Example: proper scoring rules

A decision problem $S(a, \theta)$ with $A = \Delta_\theta$ is a **proper scoring rule** if $a^*(q) = q$, i.e. it is always optimal to choose one's true belief.

Solution: take any convex function G .

For each q , there will be a tangent hyperplane.

Strictly convex \longleftrightarrow *strictly proper*
(truthfulness is uniquely optimal)



Example: proper scoring rules

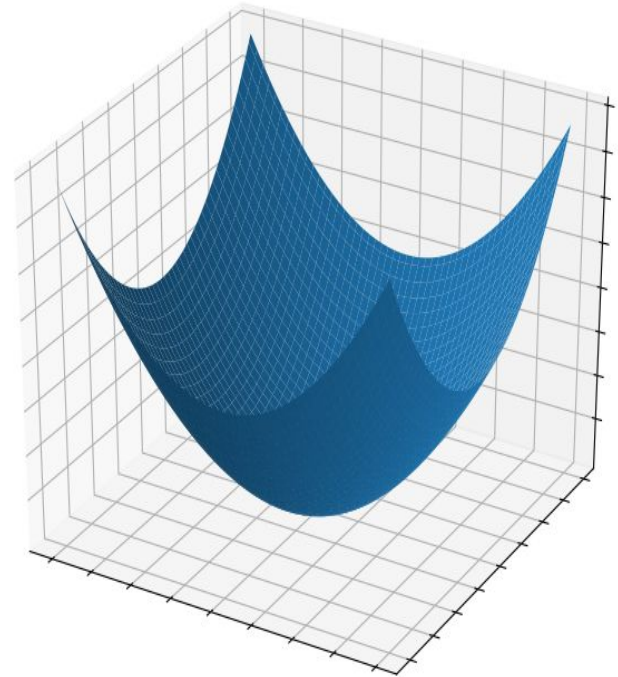
A decision problem $S(a, \theta)$ with $A = \Delta_\theta$ is a **proper scoring rule** if $a^*(q) = q$, i.e. it is always optimal to choose one's true belief.

Solution: take any convex function G .

For each q , there will be a tangent hyperplane.

Example: $S(a, \theta) = \log a(\theta)$.

- $S(a ; q) = \sum_\theta q(\theta) \log a(\theta)$.
- Optimal action $a = q$.
- $G(q) = \sum_\theta q(\theta) \log q(\theta) = -H(q)$.



Revelation principle for agents

Fact 3

For every decision problem (A, θ, u) , there is a corresponding proper scoring rule $(\Delta_\theta, \theta, S)$ that is utility-equivalent: the expected utility is always equal.

Proof.

Define $S(q, \theta) = u(a^*(q), \theta)$.

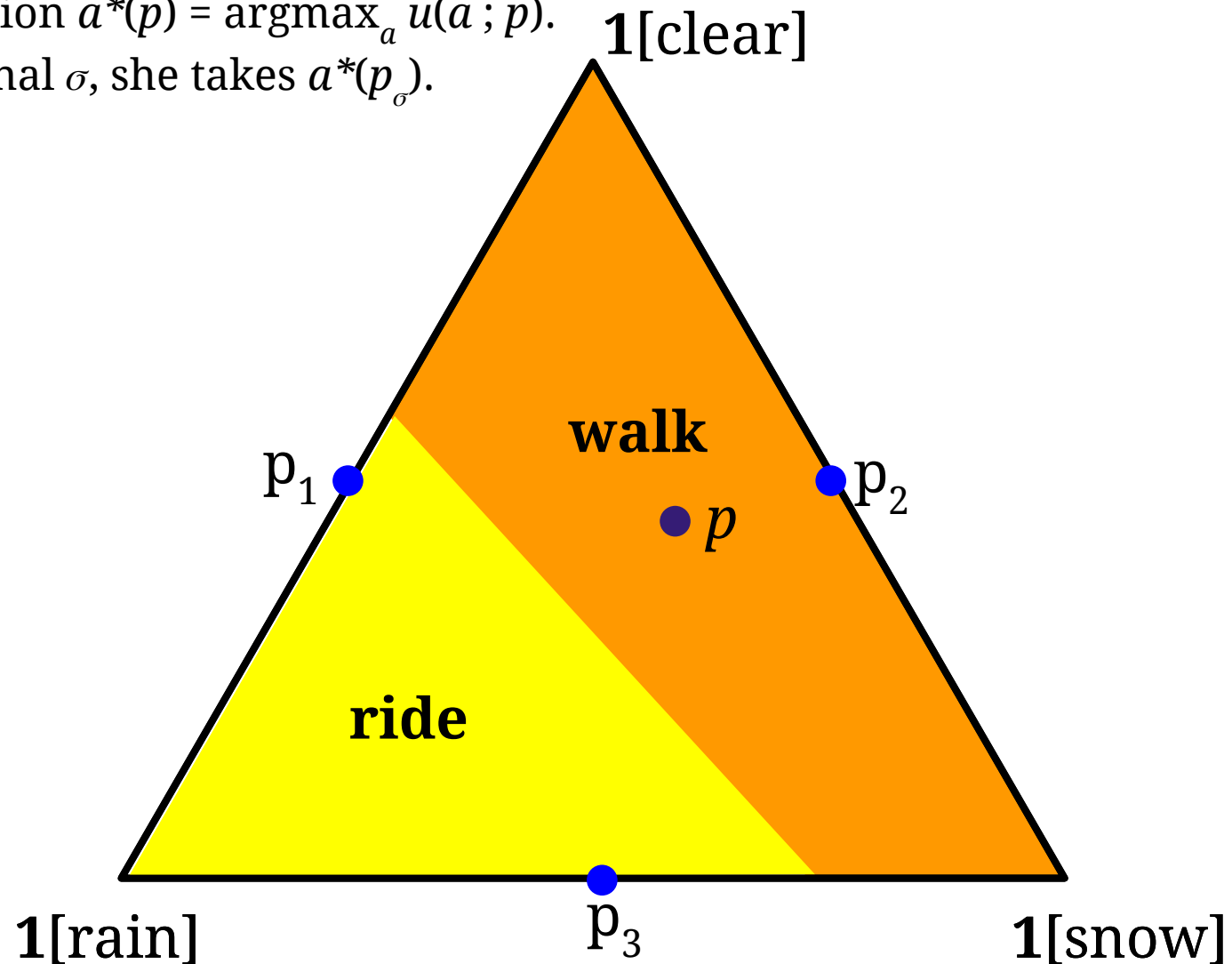
In other words: given agent's reported belief q , plug action $a^*(q)$ into the original decision problem.

Truthfulness is an optimal action, and expected utility for any belief is equal in both problems.

Signals and decisionmaking

How signals affect decisionmaking

- Agent begins with prior belief p on θ .
- She would take action $a^*(p) = \operatorname{argmax}_a u(a; p)$.
- After receiving signal σ , she takes $a^*(p_\sigma)$.



How signals affect decisionmaking

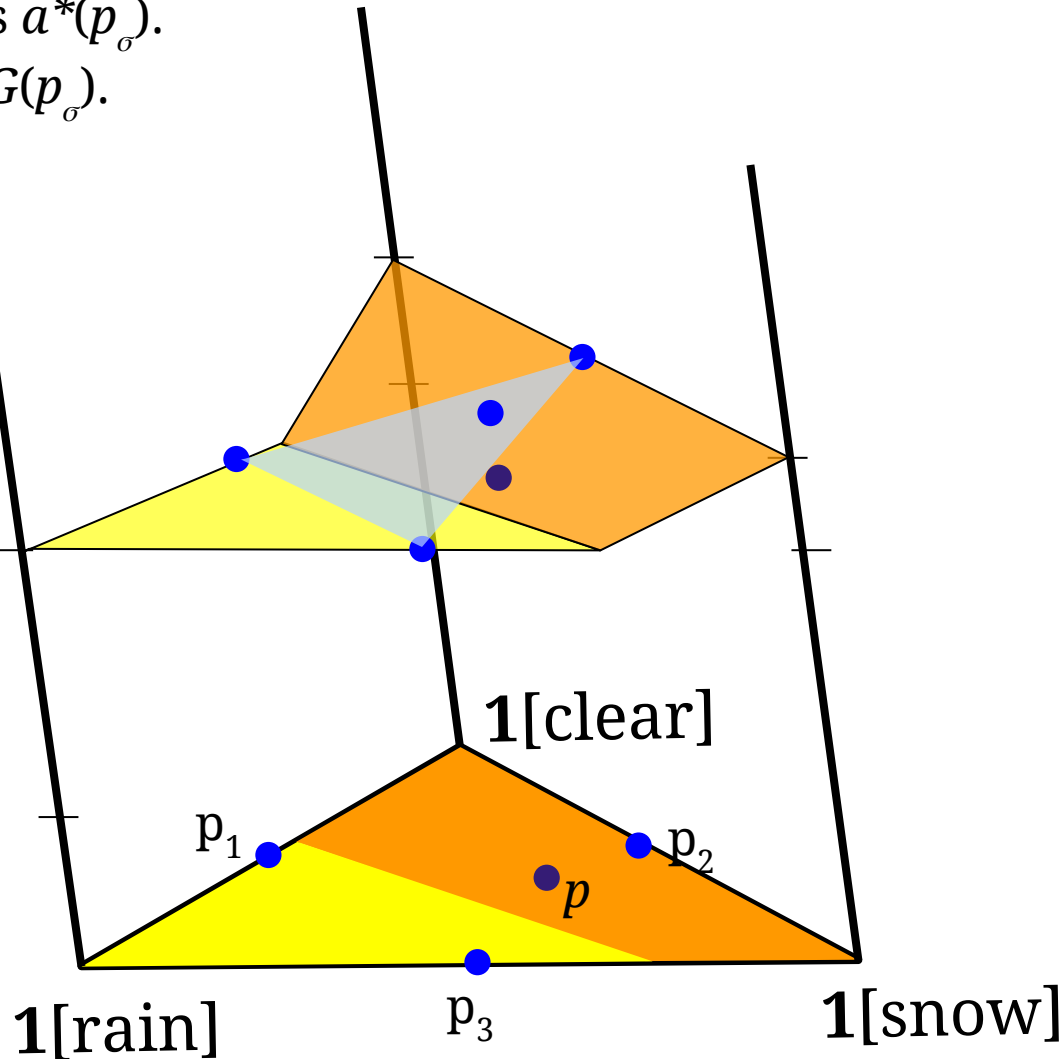
- Agent begins with prior belief p on θ .
- She would take action $a^*(p) = \operatorname{argmax}_a u(a; p)$.
- After receiving signal σ , she takes $a^*(p_\sigma)$.
- Expected utility is $V^{u,\varphi}(\Sigma) := \mathbf{E}_{\sigma \sim \varphi} G(p_\sigma)$.

Fact 4

More information always increases expected utility in a decision problem.

Proof.

$$\begin{aligned} & V^{u,\varphi}(\Sigma) \\ &= \mathbf{E}_\sigma G(p_\sigma) \\ &\geq G(\mathbf{E}_\sigma p_\sigma) \quad (\text{Jensen's inequality}) \\ &= G(p) \\ &= \text{exp. utility with no signal.} \end{aligned}$$



Revelation principle for signallers

φ is **direct** if $\Sigma = A$ (each signal *recommends* a unique action),
and **persuasive** if it is optimal to comply, i.e. $a^*(p_a) = a$.

Fact 5

For every signalling scheme φ in decision problem (A, θ, u) , there is a direct, persuasive φ' that outcome-equivalent: induces the same distribution on a, θ .

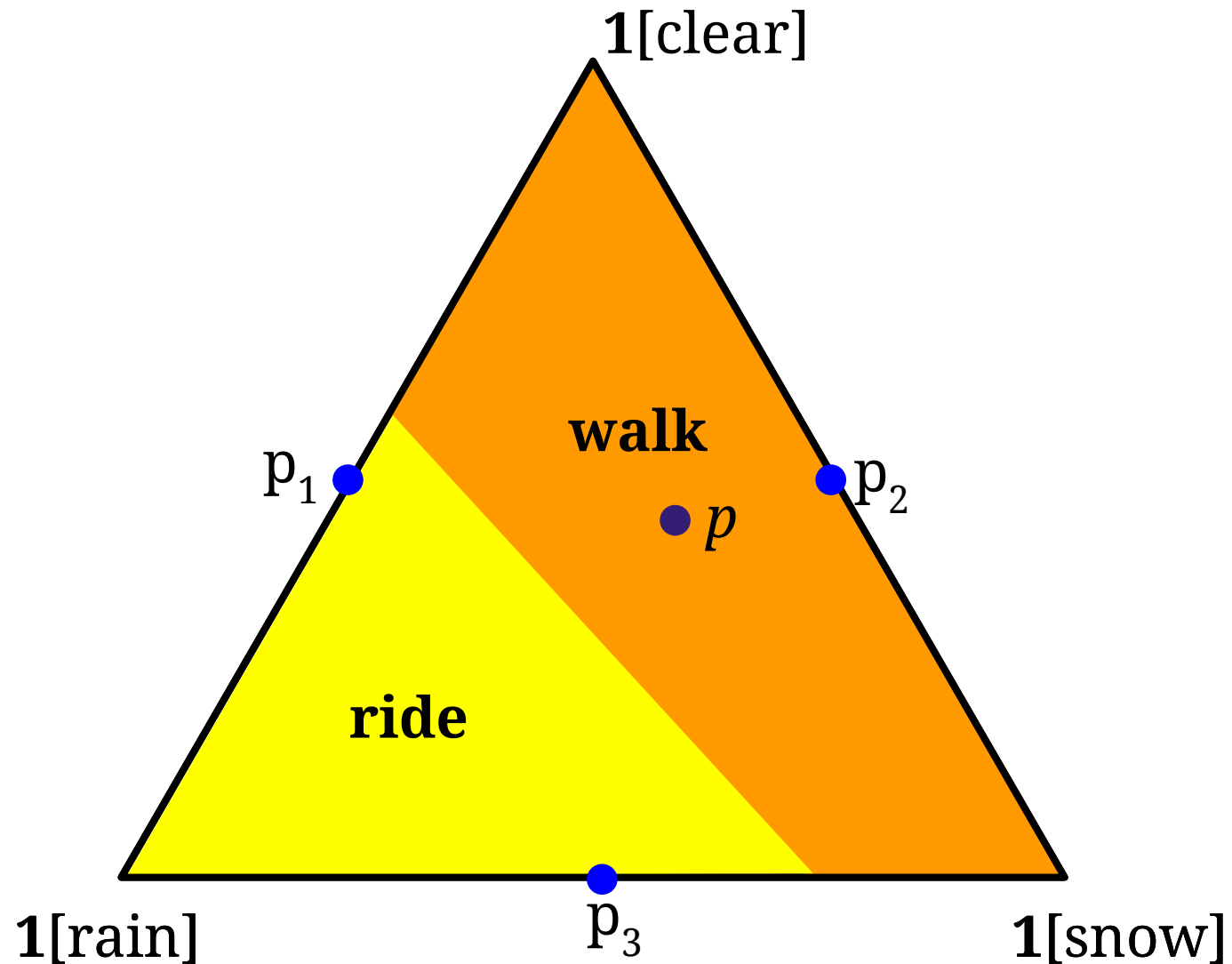
Proof.

Merge all signals σ inducing action a , i.e. $a = a^*(p_\sigma)$, into a single signal s .

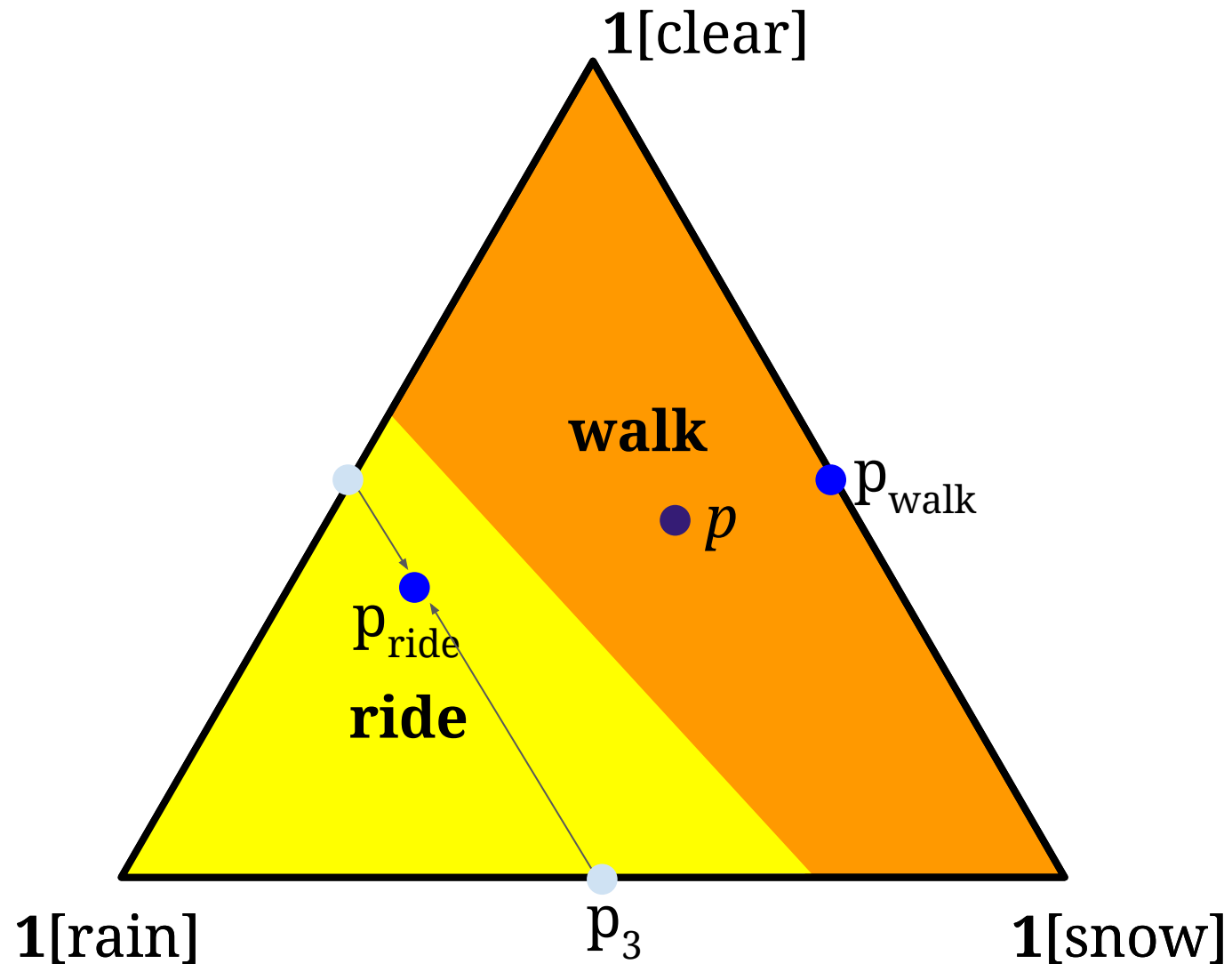
Then p_s is a convex combination of the $\{p_\sigma\}$, so
$$\operatorname{argmax}_a u(a'; p_s) = \operatorname{argmax}_a \sum_\sigma \alpha_\sigma u(a'; p_\sigma) = a.$$

Repeat for all actions.

Illustrating the signaller's revelation principle



Illustrating the signaller's revelation principle



The Blackwell order

Garblings and Blackwell order

- We have a signal Σ distributed according to $\varphi(\sigma, \theta)$.
- And we have Σ' distributed according to $\varphi'(\sigma', \theta)$.

Say Σ' is a **garbling** of Σ if it can be simulated given Σ , i.e. σ' is distributed as a randomized function $f(\sigma)$. (Σ' is conditionally independent of θ given Σ .)

Fact (cf. 1.2): Σ' is a garbling of Σ if and only if each $p_{\sigma'} = \mathbf{E}[p_{\sigma} \mid \sigma']$.

Theorem (Blackwell 1953):

Σ' is a garbling of Σ if and only if, for all decision problems, $V^{u,\varphi}(\Sigma) \geq V^{u,\varphi}(\Sigma')$.

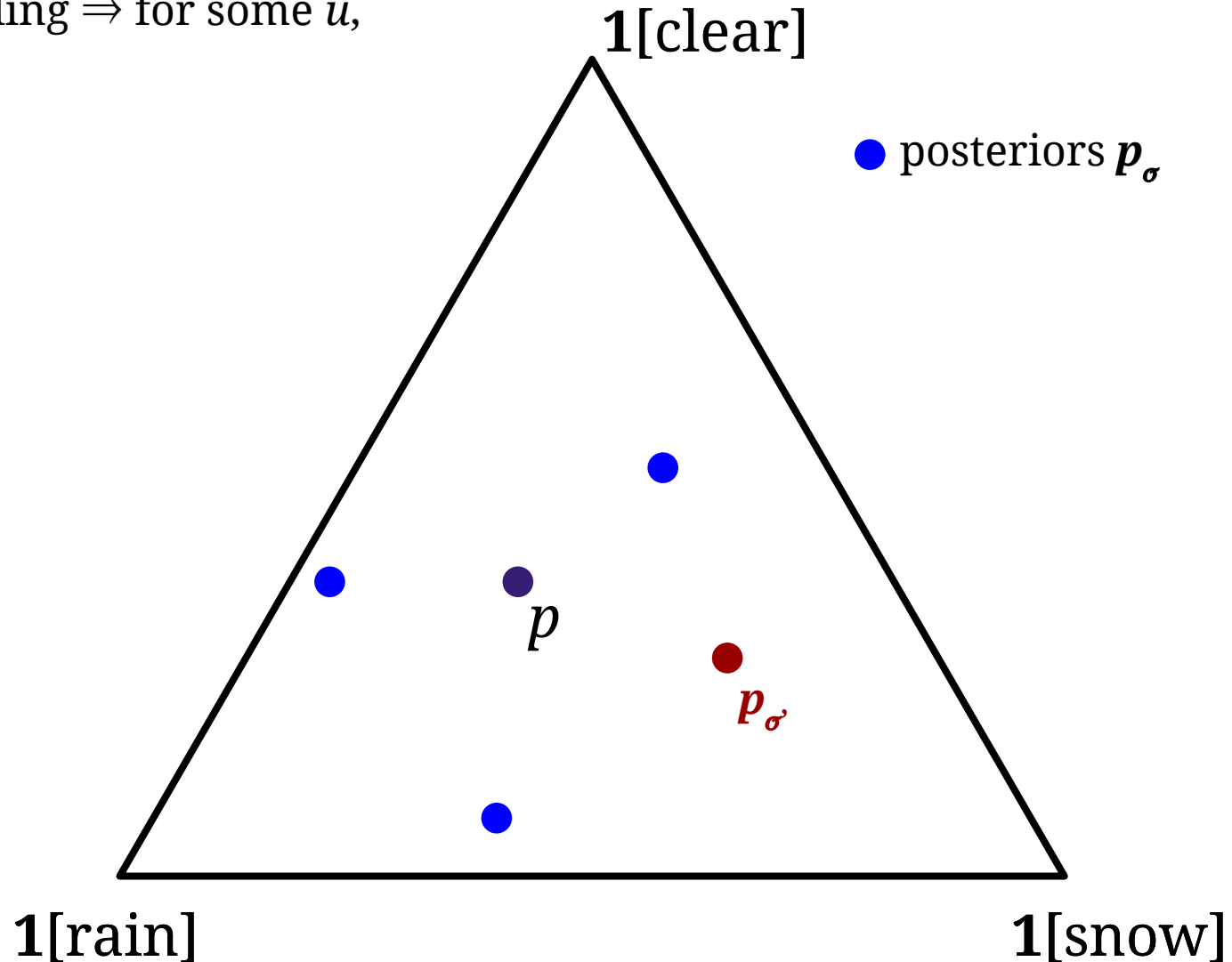
Proof sketch:

(\rightarrow) Given Σ , we can simulate a draw from Σ' and take the optimal action.

(\leftarrow) If not a garbling, there exists a realization σ' that is more informative in some direction than Σ is on average. Make a two-action decision problem that rewards this knowledge...

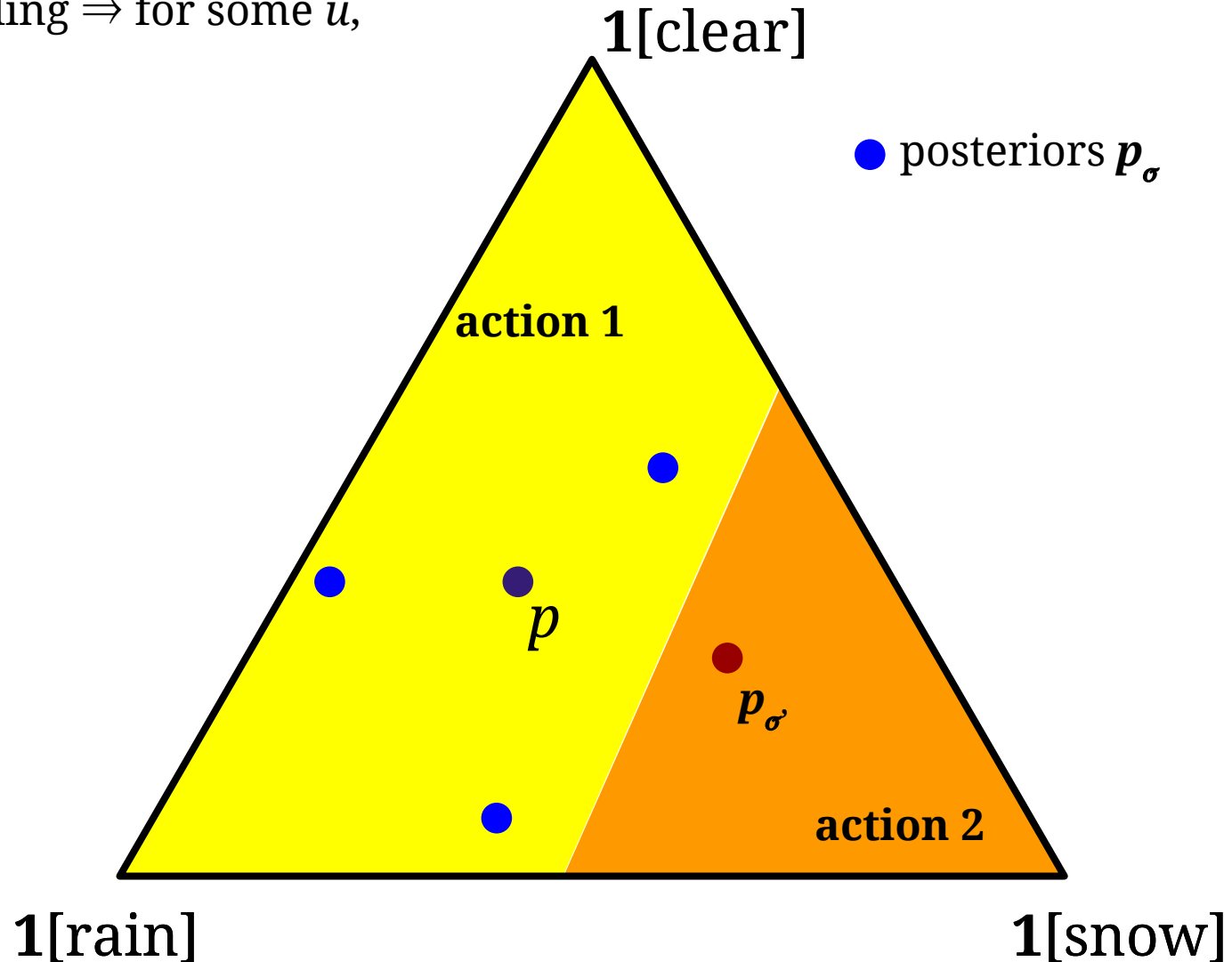
Blackwell proof - an easy version

If p_σ is not in the convex hull of $\{p_\sigma\}$, then definitely not a garbling \Rightarrow for some u , Σ' is preferable.



Blackwell proof - an easy version

If p_σ is not in the convex hull of $\{p_\sigma\}$, then definitely not a garbling \Rightarrow for some u , Σ' is preferable.

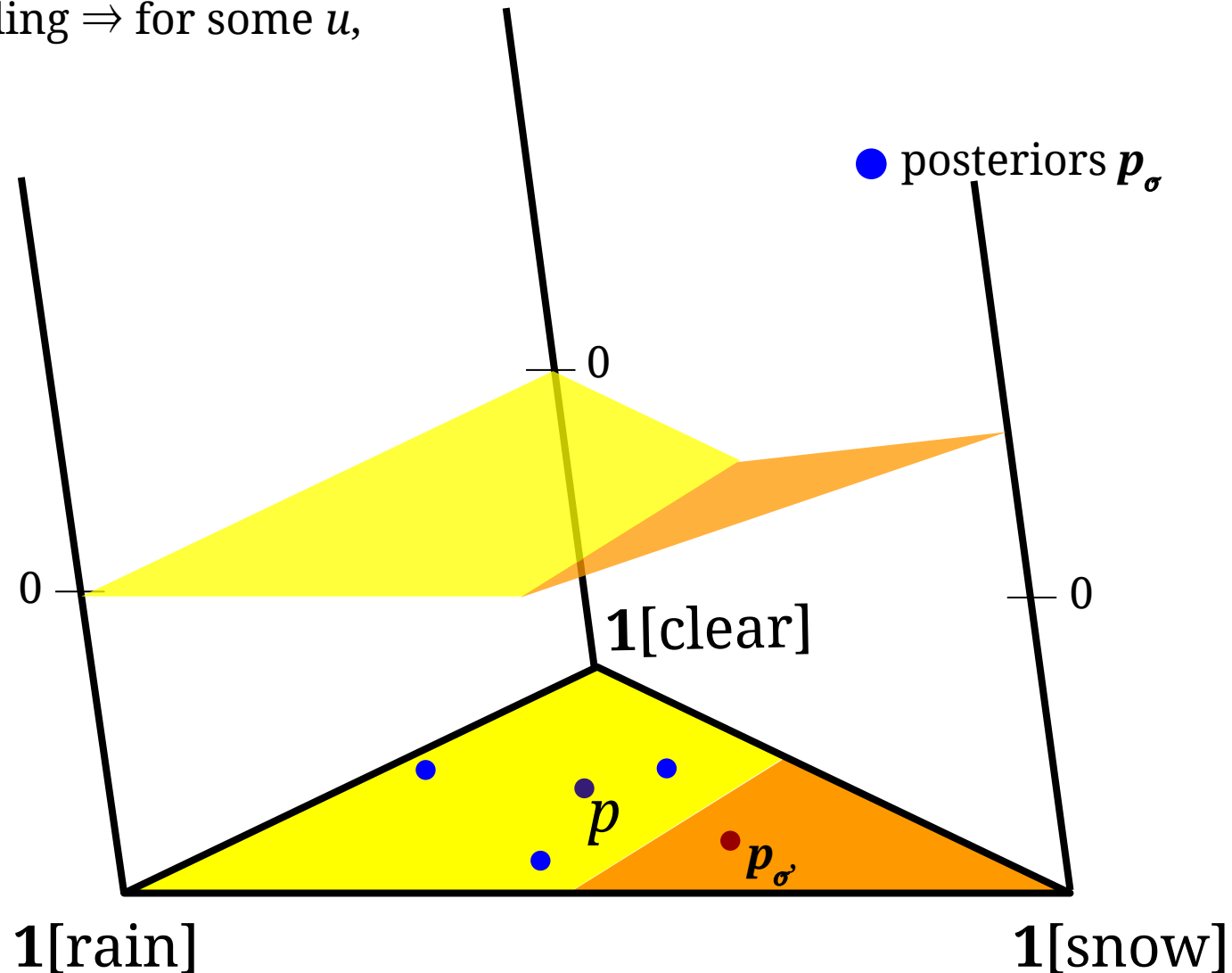


Blackwell proof - an easy version

If p_{σ} is not in the convex hull of $\{p_{\sigma}\}$, then definitely not a garbling \Rightarrow for some u , Σ' is preferable.

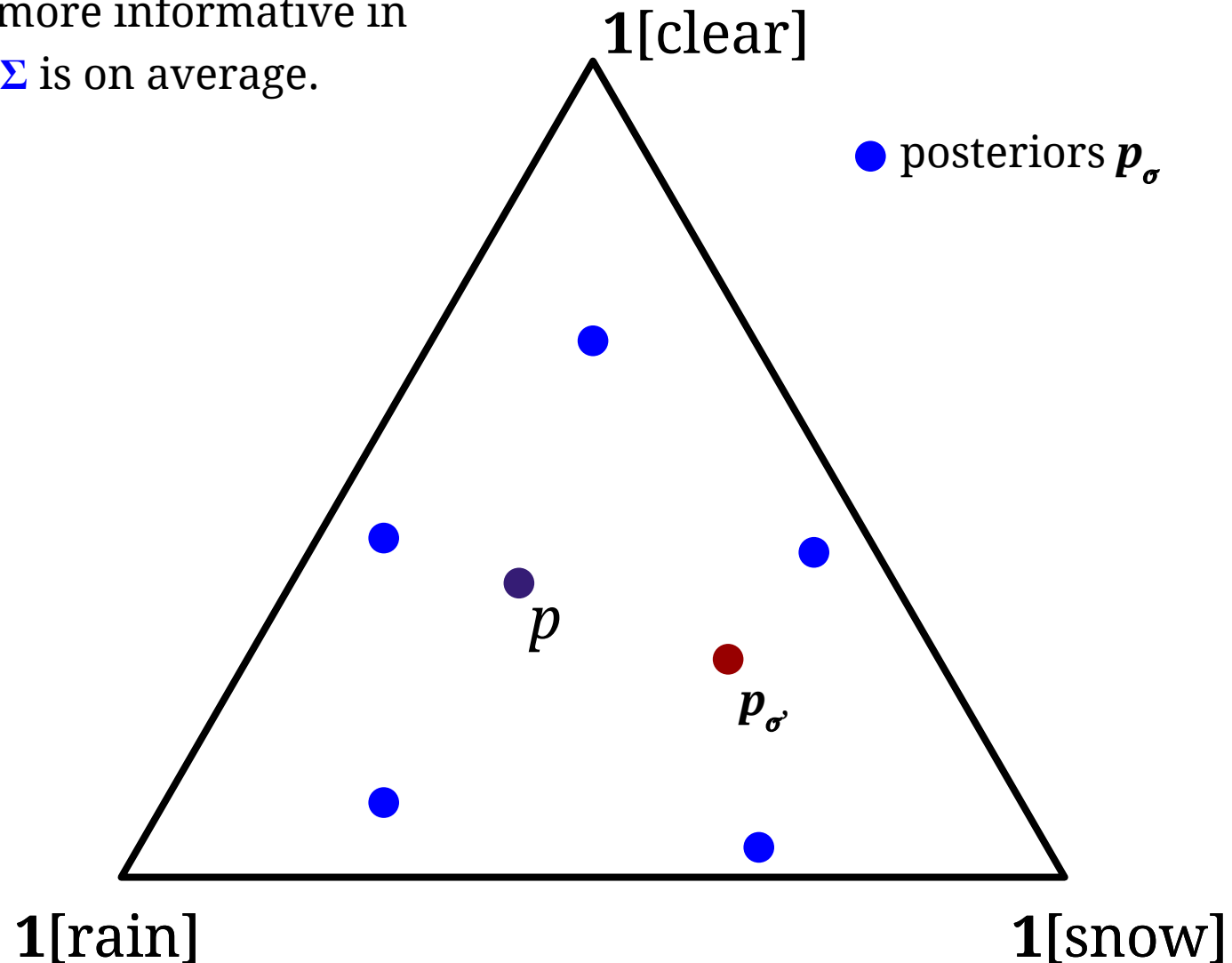
$$V^{u,\varphi}(\Sigma) = 0$$

$$V^{u,\varphi}(\Sigma') > 0.$$



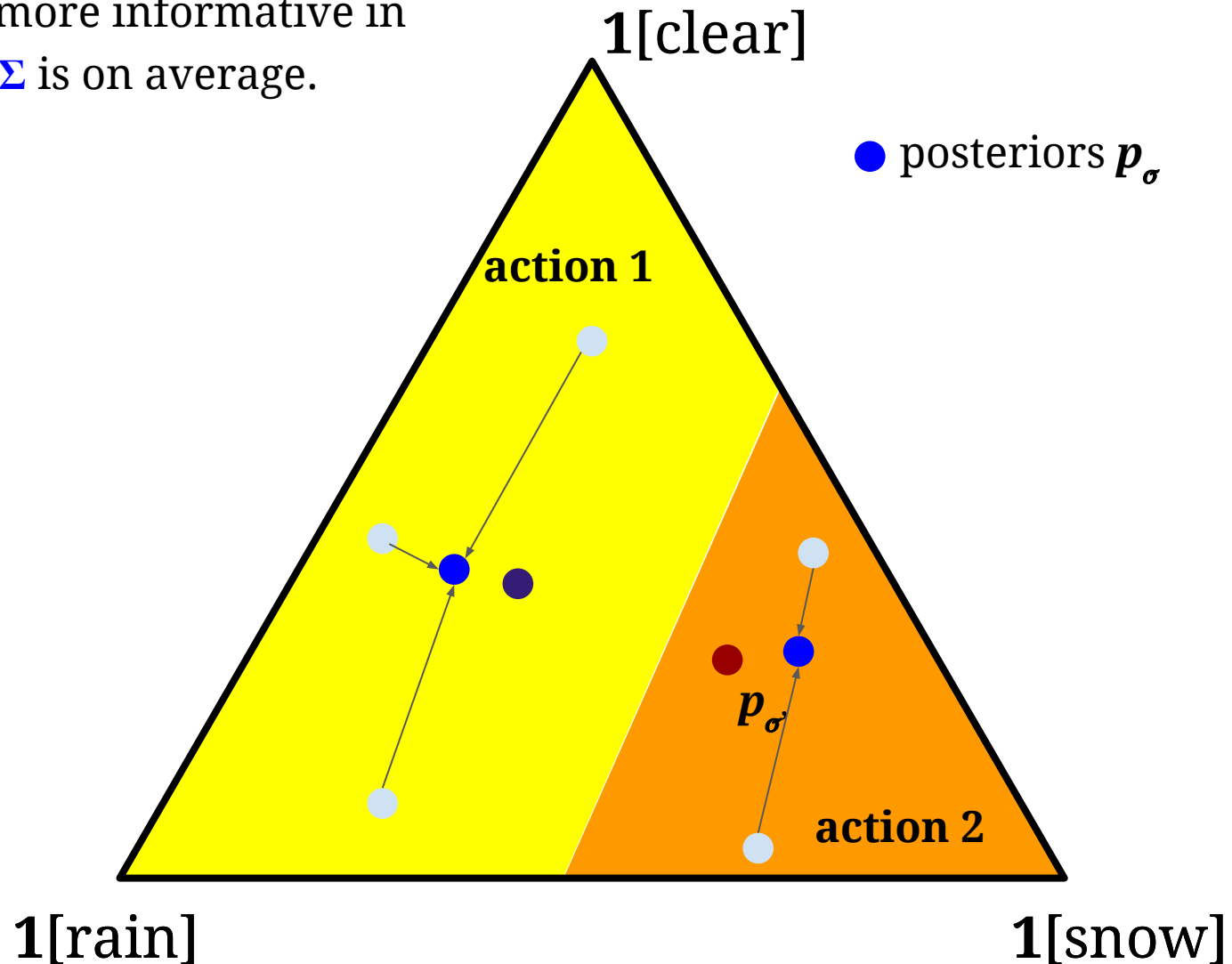
Blackwell proof - general idea

If not a garbling, there exists a realization σ' that is more informative in some direction than Σ is on average.



Blackwell proof - general idea

If not a garbling, there exists a realization σ' that is more informative in some direction than Σ is on average.



Part 1C: Bayesian games

The Game Model

The basic game

- $I = \{1, \dots, n\}$ set of agents Agent i
- A_i set of actions for agent i Agent i chooses $a_i \in A_i$
- $A = A_1 \times A_2 \dots A_n$ $a \in A$ is an **action profile**
- $\theta \in \Theta$ state of nature with prior p
- $u_i(a_i, a_{-i}, \theta)$ utility function for i

Information structure (Σ, φ) of the game

- Σ_i set of signals for i agent i observes $\sigma_i \in \Sigma_i$
- $\Sigma = \Sigma_1 \times \Sigma_2 \dots \Sigma_n$ $\sigma \in \Sigma$ is a **signal profile**
- $\varphi(\sigma, \theta)$ prob. of $\sigma \in \Sigma$ given state θ

Note: a_{-i} denotes the set of all agents' actions except i 's
(similar definition for σ_{-i})

Equilibrium Concepts

Bayes Nash Equilibrium (BNE)

- A **strategy** for player i is $\beta_i : \Sigma_i \rightarrow \Delta(A_i)$
- $\beta_i(a_i | \sigma_i) = \text{Prob}(\text{take } a_i \text{ when observing } \sigma_i)$

$\{\beta_i\}_{i=1,\dots,n}$ forms a BNE if **unilateral deviation is not beneficial** for any agent. Specifically, for any agent i , signal $\sigma_i \in \Sigma_i$, action $a_i \in A_i$ with $\beta_i(a_i | \sigma_i) > 0$,

$$\begin{aligned} & \sum_{\sigma_{-i}, a_{-i}} p(\theta) \varphi((\sigma_i, \sigma_{-i}), \theta) \left(\prod_{j \neq i} \beta_j(a_j | \sigma_j) \right) u_i((a_i, a_{-i}), \theta) \\ & \geq \sum_{\sigma_{-i}, a_{-i}} p(\theta) \varphi((\sigma_i, \sigma_{-i}), \theta) \left(\prod_{j \neq i} \beta_j(a_j | \sigma_j) \right) u_i((a'_i, a_{-i}), \theta) \end{aligned}$$

for all $a'_i \in A_i$

Equilibrium Concepts

Bayes Correlated Equilibrium (BCE)

- An action recommendation rule $\pi : \Theta \times \Sigma \rightarrow \Delta(A)$
- $\pi(a | \theta, \sigma) = \text{Prob}(\text{recommend action profile } a \text{ conditioned on } \theta, \sigma)$

π is a BCE if the **recommendation satisfies following obedience constraints:**

for any agent i , signal $\sigma_i \in \Sigma_i$, action $a_i \in A_i$,

$$\begin{aligned} & \sum_{\sigma_{-i}, a_{-i}} p(\theta) \varphi((\sigma_i, \sigma_{-i}), \theta) \pi[(a_i, a_{-i}) | \theta, (\sigma_i, \sigma_{-i})] u_i((a_i, a_{-i}), \theta) \\ & \geq \sum_{\sigma_{-i}, a_{-i}} p(\theta) \varphi((\sigma_i, \sigma_{-i}), \theta) \pi[(a_i, a_{-i}) | \theta, (\sigma_i, \sigma_{-i})] u_i((a'_i, a_{-i}), \theta) \end{aligned}$$

for all $a'_i \in A_i$

A Simple Fact

Fact 6

Any BNE corresponds to a BCE.

Proof.

Follows from definition.

Formally, let $\pi(a|\theta, \sigma) = \prod_i \beta_i(a_i|\sigma_i)$ for each θ .

Comparison of Information Structures

- Goal: compare informativeness of information structures
- Recall: Blackwell order compares informativeness of signaling schemes
 - (Σ', φ') is a **garbling** of (Σ, φ) if they can be coupled such that Σ' is independent of θ conditioned on Σ

A generalization of **garbling** for $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in \Sigma$:

Information structure (Σ, φ) is **individually sufficient** for (Σ', φ') if they can be coupled such that for any $i = 1, 2, \dots, n$, Σ'_i is independent of θ **and** Σ_{-i} conditioned on Σ_i .

Intuitively, (Σ, φ) is more informative than (Σ', φ')

Comparison of Information Structures

Theorem [Bergemann/Morris 2016]

(Σ, φ) is **individually sufficient** for (Σ', φ') if and only if the set of BCE induced by (Σ, φ) is a **subset** of the set of BCE induced by (Σ', φ') for all Bayesian games.

Remarks:

- More information means smaller BCE set (because more constraints)

Obedience constraints in BCE: for any agent i , **signal** $\sigma_i \in \Sigma_i$, action $a_i \in A_i$,

$$\begin{aligned} & \sum_{\sigma_{-i}, a_{-i}} p(\theta) \varphi((\sigma_i, \sigma_{-i}), \theta) \pi[(a_i, a_{-i}) | \theta, (\sigma_i, \sigma_{-i})] u_i((a_i, a_{-i}), \theta) \\ & \geq \sum_{\sigma_{-i}, a_{-i}} p(\theta) \varphi((\sigma_i, \sigma_{-i}), \theta) \pi[(a_i, a_{-i}) | \theta, (\sigma_i, \sigma_{-i})] u_i((a'_i, a_{-i}), \theta) \end{aligned}$$

for all $a'_i \in A_i$

Comparison of Information Structures

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Remarks:

- More information means smaller BCE set (because more constraints)
- Defines a *partial order* over information structures

Prisoner's Dilemma of Incomplete Information

	Cooperate	Defect
Cooperate	-3 -3	0 -9
Defect	-9 0	-6 -6

Prisoner's Dilemma of Incomplete Information

	Cooperate	Defect	
Cooperate	$-3 + \theta$ $-3 + \theta$	0 $-9 + \theta$	Reward $\theta \sim \text{uniform}\{0, 2, 4.1\}$ to encourage cooperation
Defect	$-9 + \theta$ 0	-6 -6	

- Information structure I: players know θ exactly (full information)
 - A game of complete information is played for each θ
 - Unique BNE and BCE

Prisoner's Dilemma of Incomplete Information

	Cooperate	Defect
Cooperate	$-3 + \theta$ $-3 + \theta$	0 $-9 + \theta$
Defect	0 $-9 + \theta$	-6 -6

Reward $\theta \sim \text{uniform}\{0, 2, 4.1\}$
to encourage cooperation

- Information structure II: players know whether $\theta = 0$ or not
 - After realization of θ , φ reveals “ $\theta \neq 0$ ” or “ $\theta = 0$ ”
- Unique Bayes Nash Equilibrium:
 - For each player: “ $\theta = 0$ ” \rightarrow defect; “ $\theta \neq 0$ ” \rightarrow cooperate

Prisoner's Dilemma of Incomplete Information

	Cooperate	Defect
Cooperate	$-3 + \theta$ $-3 + \theta$	0 $-9 + \theta$
Defect	$-9 + \theta$ 0	-6 -6

Reward $\theta \sim \text{uniform}\{0, 2, 4.1\}$
to encourage cooperation

- Information structure II: players know whether $\theta = 0$ or not
 - After realization of θ , φ reveals “ $\theta \neq 0$ ” or “ $\theta = 0$ ”
- This BNE corresponds to a BCE, but there are also other BCEs.
 - $\pi(\text{cooperate, cooperate} \mid \theta = 4.1, \text{“}\theta \neq 0\text{”}) = 1$
 - $\pi(\text{defect, defect} \mid \theta = 2, \text{“}\theta \neq 0\text{”}) = 1$
 - $\pi(\text{defect, defect} \mid \theta = 0, \text{“}\theta = 0\text{”}) = 1$

Prisoner's Dilemma of Incomplete Information

	Cooperate	Defect	
Cooperate	$-3 + \theta$ $-3 + \theta$	0 $-9 + \theta$	Reward $\theta \sim \text{uniform}\{0, 2, 4.1\}$ to encourage cooperation
Defect	0 $-9 + \theta$	-6 -6	

- Information structure III: players know nothing about θ besides its prior

Exercise: prove there is still a unique BNE, but the set of BCE is even larger than the previous one

A Useful Remark for Bayesian Games

A Bayesian agent
in equilibrium
cannot be misinformed

(There is no lying or misinformation.
Agents can only be more informed or less informed)

Explanation: models assume that players know all prior distributions, and player actions and signals are indeed drawn from these distributions.

Recap and Takeaways

Signals

- Signal \longleftrightarrow set of posterior beliefs $\{p_1, \dots, p_n\}$ whose expectation is the prior.
(amenable to linear programs)

Decision problems

- Decision problem \longleftrightarrow convex function G on Δ_θ .

Revelation principles

- For agents: can WLOG report belief on θ ; optimal action is simulated.
- For signaller: can WLOG signal a recommended action (it will be followed).

Bayesian Games

- Games with uncertainty and player private information
- Main solution concepts: Bayes-Nash and Bayes-Correlated equilibria
- Information structure affects equilibrium

Notation

- A set of actions agent chooses a
 - Θ set of states of the world nature draws θ
 - $u(a, \theta)$ utility function
-
- p prior distribution on Θ known to agent
 - Σ a signal (also refers to set of realizations) agent observes $\Sigma=\sigma$
 - $\varphi(\sigma, \theta)$ probability of signal σ given state θ
 - p_σ posterior distribution on Θ given σ given by Bayes' rule
-
- $u(a ; q) = \mathbf{E}_{\theta \sim q} u(a, \theta)$ linear function of q
 - $G(q) = \max_a u(a ; q)$ convex function of q
 - $a^*(q) = \operatorname{argmax}_a u(a ; q)$ optimal action given q
 - $V^{u, \varphi}(\Sigma) = \mathbf{E}_{\sigma \sim \varphi} G(p_\sigma)$ exp. utility observing Σ