Supplemental Material: Controlling Elections through Social Influence

1 Missing proofs

We now provide proofs that were deferred from the main text. We start out with with the full proofs for the MOV_C and MOV_D objectives:
Theorem 4.1: *In an election with two candidates, MOV is a monotone submodular function.*

Proof. We first fix a particular scenario y and show that the function $f(\cdot, y, V_{c*}^2)$ is submodular. This suffices to show that $\mathbb{E}_y[f(\cdot, y, V_{c*}^2)]$ is submodular since a nonnegative linear combination of submodular functions remains submodular. Monotonicity is clear since adding additional seeds to A can only make more nodes reachable. To show submodularity, we can write the marginal gain as

$$f(A \cup \{x\}, y, V_{c*}^2) - f(A, y, V_{c*}^2) = \sum_{v \in V_{c*}^2} (1 - \chi(v, A, y))\chi(v, \{x\}, y).$$

Compare the above expression for a set A and any $B \supseteq A$. For any single node v, $\chi(v, B, y) = 1$ whenever $\chi(v, A, y) = 1$. Hence, the term in the above summation for each node v can only be smaller for $f(B \cup \{x\}, y, V_{c*}^2) - f(B, y, V_{c*}^2)$ than for $f(A \cup \{x\}, y, V_{c*}^2) - f(A, y, V_{c*}^2)$. We conclude that $f(A \cup \{x\}, y, V_{c*}^2) - f(A, y, V_{c*}^2) \geq f(B \cup \{x\}, y, V_{c*}^2) - f(B, y, V_{c*}^2)$ and submodularity now follows by taking the expectation over y. □

Theorem 5.2: MOVConstructive *obtains a* $\frac{1}{3}\left(1 - \frac{1}{e}\right)$*-approximation to the* MOV_C *problem with any number of candidates.*

Proof. Let $c(S, y) = \arg\min_{c_i} f(V_{c*}^2 \cap V_{c_i}^1) - |V_{c_i}^1|$ be the candidate achieving the minimum in the definition of m_C. Let S^* be an optimal seed set. Note that for all scenarios y, seed sets S, and candidates c_i, $f(S, y, V_{c*}^2) \geq f(S, y, V_{c*}^2 \cap V_{c_i}^1)$. Hence, we have

$$\begin{aligned}\mathop{\mathbb{E}}_y\left[f\left(S^*, y, V_{c*}^2\right)\right] \geq \frac{1}{3}\mathop{\mathbb{E}}_y\Big[&f\left(S^*, y, V_{c*}^1\right) + f\left(S^*, y, V_{c*}^1 \cap V_{c(S^*,y)}^1\right)\\ &+ f\left(S^*, y, V_{c*}^1 \cap V_{c(S,y)}^1\right)\Big]\end{aligned}$$

Note that $\mathbb{E}_y[f(\cdot, y, V_{c*}^2)]$ is a monotone submodular function, which MOVConstructive greedily maximizes. Let S be the resulting seed set. We have

$$
\begin{aligned}
&\mathop{\mathbb{E}}_{y}\left[f\left(S, y, V_{c_*}^2\right) + f\left(S, y, V_{c_*}^2 \cap V_{c(S,y)}^1\right)\right] \\
&\geq \mathop{\mathbb{E}}_{y}\left[f\left(S, y, V_{c_*}^2\right)\right] \\
&\geq \frac{1}{3}\left(1-\frac{1}{e}\right)\mathop{\mathbb{E}}_{y}\left[f\left(S^*, y, V_{c_*}^2\right) + f\left(S^*, y, V_{c_*}^2 \cap V_{c(S^*,y)}^1\right)\right. \\
&\left. + f\left(S^*, y, V_{c_*}^2 \cap V_{c(S,y)}^1\right)\right]
\end{aligned}
$$

and so

$$
\begin{aligned}
&\mathrm{MOV}_C(S) \\
&= \mathop{\mathbb{E}}_{y}\left[f\left(S, y, V_{c_*}^2\right) + \min_{c_j} f\left(S, y, V_{c_*}^2 \cap V_{c_j}^1\right) + \max_{c_i}|V_{c_i}^1| - |V_{c_j}^1|\right] \\
&= \mathop{\mathbb{E}}_{y}\left[f\left(S, y, V_{c_*}^2\right) + f\left(S, y, V_{c_*}^2 \cap V_{c(S,y)}^1\right)\right] + \max_{c_i}|V_{c_i}^1| - \mathop{\mathbb{E}}_{y}\left[|V_{c(S,y)}^1|\right] \\
&\geq \frac{1}{3}\left(1-\frac{1}{e}\right)\mathop{\mathbb{E}}_{y}\left[f\left(S^*, y, V_{c_*}^1\right) + f\left(S^*, y, V_{c_*}^2 \cap V_{c(S^*,y)}^1\right)\right. \\
&\qquad\qquad \left. + f\left(S^*, y, V_{c_*}^2 \cap V_{c(S,y)}^1\right)\right] + \max_{c_i}|V_{c_i}^1| - \mathop{\mathbb{E}}_{y}\left[|V_{c(S,y)}^1|\right] \\
&\geq \frac{1}{3}\left(1-\frac{1}{e}\right)\mathop{\mathbb{E}}_{y}\left[f\left(S^*, y, V_{c_*}^1\right) + f\left(S^*, y, V_{c_*}^2 \cap V_{c(S^*,y)}^1\right)\right. \\
&\qquad\qquad \left. + f\left(S^*, y, V_{c_*}^1 \cap V_{c(S,y)}^1\right) + \max_{c_i}|V_{c_i}^1| - |V_{c(S,y)}^1|\right] \\
&= \frac{1}{3}\left(1-\frac{1}{e}\right)\mathop{\mathbb{E}}_{y}\left[f\left(S^*, y, V_{c_*}^1\right) + f\left(S^*, y, V_{c_*}^2 \cap V_{c(S^*,y)}^1\right)\right. \\
&\left. + f\left(S^*, y, V_{c_*}^1 \cap V_{c(S,y)}^1\right) + \max_{c_i}|V_{c_i}^1| - |V_{c(S,y)}^1| + |V_{c(S^*,y)}^1| - |V_{c(S^*,y)}^1|\right] \\
&= \frac{1}{3}\left(1-\frac{1}{e}\right)\mathop{\mathbb{E}}_{y}\left[f\left(S^*, y, V_{c_*}^1\right) + \min_{c_j}\left(f(S^*, y, V_{c_*}^2 \cap V_{c_j}^1) - |V_{c_j}^1|\right)\right. \\
&\qquad\qquad \left. + f(S^*, y, V_{c_*}^2 \cap V_{c(S,y)}^1) + \max_{c_i}|V_{c_i}^1| - |V_{c(S,y)}^1| + |V_{c(S^*,y)}^1|\right] \\
&= \frac{1}{3}\left(1-\frac{1}{e}\right)\left(\mathrm{MOV}_C(S^*) + \mathop{\mathbb{E}}_{y}\left[f\left(S^*, y, V_{c_*}^2 \cap V_{c(S,y)}^1\right)\right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left. + |V_{c(S^*,y)}^1| - |V_{c(S,y)}^1|\right]\right).
\end{aligned}
$$

Now by definition of $c(S^*, y)$, $f(S^*, y, V_{c_*}^2 \cap V_{c(S^*,y)}^1) - |V_{c(S^*,y)}^1| \leq f(S^*, y, V_{c_*}^2 \cap V_{c(S,y)}^1) - |V_{c(S,y)}^1|$ and so

$$
|V_{c(S^*,y)}^1| - |V_{c(S,y)}^1| \geq f\left(S^*, y, V_{c_*}^2 \cap V_{c(S^*,y)}^1\right) - f\left(S^*, y, V_{c_*}^2 \cap V_{c(S,y)}^1\right)
$$

This yields

$$
\mathrm{MOV}_C(S) \geq \frac{1}{3}\left(1-\frac{1}{e}\right)\left(\mathrm{MOV}_C(S^*) + \mathop{\mathbb{E}}_{y}\left[f\left(S^*, y, V_{c_*}^2 \cap V_{c(S,y)}^1\right) + \right.\right.
$$

$$f\left(S^*, y, V_{c_*}^2 \cap V_{c(S^*,y)}^1\right) - f\left(S^*, y, V_{c_*}^2 \cap V_{c(S,y)}^1\right)\Big]\Bigg)$$

$$= \frac{1}{3}\left(1 - \frac{1}{e}\right)\left(\mathrm{MOV}_C(S^*) + \mathop{\mathbb{E}}_y\left[f\left(S^*, y, V_{c_*}^2 \cap V_{c(S^*,y)}^1\right)\right]\right)$$

$$\geq \frac{1}{3}\left(1 - \frac{1}{e}\right)\mathrm{MOV}_C(S^*).$$

□

Theorem 5.3: MOVDestructive *obtains a* $\frac{1}{2}\left(1 - \frac{1}{e}\right)$*-approximation to the multicandidate* MOV_D *problem.*

Proof. Now let $c(S, y) = \arg\max_{c_i} f(V_{c_*}^1 \cap V_{c_i}^2) + |V_{c_i}^1|$ be the candidate achieving the maximum in the definition of m_D. Let S^* be an optimal seed set. Similarly to before, we have

$$\mathop{\mathbb{E}}_y\left[f\left(S^*, y, V_{c_*}^1\right)\right] \geq \frac{1}{2}\mathop{\mathbb{E}}_y\left[f\left(S^*, y, V_{c_*}^1\right) + f\left(S^*, y, V_{c_*}^1 \cap V_{c(S^*,y)}^2\right)\right].$$

MOVDestructive greedily maximizes $\mathbb{E}_y\left[f\left(S^*, y, V_{c_*}^1\right)\right]$. Call the resulting seed set S. We have

$$\begin{aligned}
&\mathrm{MOV}_D(S)\\
&= \mathop{\mathbb{E}}_y\left[f\left(S, y, V_{c_*}^1\right) + f\left(S, y, V_{c_*}^1 \cap V_{c(S,y)}^2\right) + |V_{c(S,y)}^1| - \max_{c_i}|V_{c_i}^1|\right]\\
&\geq \left(1 - \frac{1}{e}\right)\mathop{\mathbb{E}}_y\left[f\left(S^*, y, V_{c_*}^1\right)\right] + \mathop{\mathbb{E}}_y\Big[f\left(S, y, V_{c_*}^1 \cap V_{c(S,y)}^2\right)\\
&\qquad\qquad + |V_{c(S,y)}^1| - \max_{c_i}|V_{c_i}^1|\Big]\\
&\geq \frac{1}{2}\left(1 - \frac{1}{e}\right)\mathop{\mathbb{E}}_y\left[f\left(S^*, y, V_{c_*}^1\right) + f\left(S^*, y, V_{c_*}^1 \cap V_{c(S^*,y)}^2\right)\right]\\
&\qquad + \mathop{\mathbb{E}}_y\left[f\left(S, y, V_{c_*}^1 \cap V_{c(S,y)}^2\right) + |V_{c(S,y)}^1| - \max_{c_i}|V_{c_i}^1|\right]\\
&\geq \frac{1}{2}\left(1 - \frac{1}{e}\right)\mathop{\mathbb{E}}_y\Big[f\left(S^*, y, V_{c_*}^1\right) + f\left(S^*, y, V_{c_*}^1 \cap V_{c(S^*,y)}^2\right)\\
&\qquad + f\left(S, y, V_{c_*}^1 \cap V_{c(S,y)}^2\right) + |V_{c(S,y)}^1| - \max_{c_i}|V_{c_i}^1|\Big]\\
&\geq \frac{1}{2}\left(1 - \frac{1}{e}\right)\mathop{\mathbb{E}}_y\Big[f\left(S^*, y, V_{c_*}^1\right) + f\left(S^*, y, V_{c_*}^1 \cap V_{c(S^*,y)}^2\right)\\
&+ f\left(S, y, V_{c_*}^1 \cap V_{c(S,y)}^2\right) + |V_{c(S,y)}^1| + |V_{c(S^*,y)}^1| - |V_{c(S^*,y)}^1| - \max_{c_i}|V_{c_i}^1|\Big]\\
&\geq \frac{1}{2}\left(1 - \frac{1}{e}\right)\Big[\mathrm{MOV}_D(S^*) + \mathop{\mathbb{E}}_y\Big[f\left(S, y, V_{c_*}^1 \cap V_{c(S,y)}^2\right)\\
&\qquad\qquad + |V_{c(S,y)}^1| - |V_{c(S^*,y)}^1|\Big]\Big].
\end{aligned}$$

Now using the definition of $c(S, y)$, we have that $f(S, y, V_{c_*}^1 \cap V_{c(S,y)}^2) + |V_{c(S,y)}^1| \geq f(S, y, V_{c_*}^1 \cap V_{c(S^*,y)}^2) + |V_{c(S^*,y)}^1|$. This yields

$$|V^1_{c(S,y)}| - |V^1_{c(S^*,y)}| \geq f\left(S, y, V^1_{c_*} \cap V^2_{c(S^*,y)}\right) - f\left(S, y, V^1_{c_*} \cap V^2_{c(S,y)}\right)$$

and so we have

$$\begin{aligned}
\text{MOV}_D(S) &\geq \frac{1}{2}\left(1 - \frac{1}{e}\right)\left[\text{MOV}_D(S^*) + \mathop{\mathbb{E}}_y\left[f\left(S, y, V^1_{c_*} \cap V^2_{c(S^*,y)}\right)\right]\right] \\
&\geq \frac{1}{2}\left(1 - \frac{1}{e}\right)\text{MOV}_D(S^*)
\end{aligned}$$

□

We now prove corresponding bicriteria guarantees for the POV objectives.

Theorem 5.4: *Let $OPT(\Delta)$ denote the optimal value of the problem $\max_{|S|\leq k} Pr_y\left[m_C(S,y) \geq \Delta\right]$. Let S be the set produced by* POVConstructive. *We have*

$$POV_C(S) \geq \max_{0<\alpha<1} \frac{\frac{e-1}{3e-1} OPT\left(\frac{1}{\alpha}\Delta_C\right) - \alpha}{1-\alpha}$$

Proof. The main difference from the two candidate case is that $\frac{1}{m}\sum_y \min\left(\beta, m_C(S,y)\right)$ is no longer submodular since m_C need not be a submodular function. However, the proof of Theorem 4.3 only uses submodularity in establishing an approximation guarantee for greedy optimization of the surrogate. In the multicandidate case, we will greedily optimize $\frac{1}{m}\sum_y \min\left(\beta, f(S, y, V^2_{c_*})\right)$, which is submodular. Let S_β be the resulting seed set and S^* be a set that optimizes $\frac{1}{m}\sum_y \min\left(\beta, m_C(S,y)\right)$. If we can prove that $\frac{1}{m}\sum_y \min\left(\beta, m_C(S_\beta,y)\right) \geq \gamma\frac{1}{m}\sum_y \min\left(\beta, m_C(S^*,y)\right)$ for some constant factor γ, then the same argument as in Theorem 4.3 extends to the multicandidate case. Fix any particular value of β. We establish a constant factor approximation as follows:

$$\begin{aligned}
&\frac{1}{m}\sum_y \min\left(\beta, m_C(S^*,y)\right) \\
&= \frac{1}{m}\sum_y \min\left(\beta, m_C(S_\beta,y) + m_C(S^*,y) - m_C(S_\beta,y)\right) \\
&\leq \frac{1}{m}\sum_y \min\Big(\beta, m_C(S_\beta,y) + f\left(S^*, y, V^2_{c_*}\right) - f\left(S_\beta, y, V^2_{c_*}\right) \\
&+ f\left(S^*, y, V^2_{c_*} \cap V^1_{c(S^*,y)}\right) - f\left(S_\beta, y, V^2_{c_*} \cap V^1_{c(S_\beta,y)}\right) \\
&+ |V^1_{c(S_\beta,y)}| - |V^1_{C(S^*,y)}|\Big)
\end{aligned}$$

Via the definition of $c(S^*,y)$, we have that $|V^1_{c(S_\beta,y)}| - |V^1_{c(S^*,y)}| \leq f\left(S^*, y, V^2_{c_*} \cap V^1_{c(S_\beta,y)}\right) - f\left(S^*, y, V^2_{c_*} \cap V^1_{c(S^*,y)}\right)$. This yields

$$\frac{1}{m}\sum_y \min\left(\beta, m_C(S^*,y)\right)$$

$$\leq \frac{1}{m}\sum_y \min\left(\beta, m_C(S_\beta,y) + f\left(S^*,y,V^2_{c_*}\right) + f\left(S^*,y,V^2_{c_*}\cap V^1_{c(S_\beta,y)}\right)\right)$$
$$\leq \frac{1}{m}\sum_y \min\left(\beta, m_C(S_\beta,y) + 2f\left(S^*,y,V^2_{c_*}\right)\right)$$
$$\leq \frac{1}{m}\sum_y \min\left(\beta, m_C(S_\beta,y)\right) + \frac{2}{m}\sum_y \min\left(\beta, f\left(S^*,y,V^2_{c_*}\right)\right)$$
$$\leq \frac{1}{m}\sum_y \min\left(\beta, m_C(S_\beta,y)\right) + \frac{1}{m}\frac{2e}{e-1}\sum_y \min\left(\beta, f\left(S_\beta,y,V^2_{c_*}\right)\right)$$
$$\leq \frac{1}{m}\sum_y \min\left(\beta, m_C(S_\beta,y)\right) + \frac{1}{m}\frac{2e}{e-1}\sum_y \min\left(\beta, m_C(S_\beta,y)\right)$$
$$\leq \left(1+\frac{2e}{e-1}\right)\frac{1}{m}\sum_y \min\left(\beta, m_C(S_\beta,y)\right)$$

and now the conclusion follows by applying the same argument as in Theorem 4.3. □

Theorem 5.5: *Let $OPT(\Delta)$ denote the optimal value of the problem $\max_{|S|\leq k} Pr_y\left[m_D(S,y)\geq\Delta\right]$. Let S be the set produced by* POVDestructive. *We have*

$$POV_D(S) \geq \max_{0<\alpha<1} \frac{\frac{e-1}{3e-1}OPT(\frac{1}{\alpha}\Delta) - \alpha}{1-\alpha}$$

Proof. Applying the reasoning as in Theorem 5.4, we have

$$\frac{1}{m}\sum_y \min\left(\Delta, m_D(S^*,y)\right)$$
$$= \frac{1}{m}\sum_y \min\left(\Delta, m_C(S,y) + m_D(S^*,y) - m_D(S,y)\right)$$
$$\leq \frac{1}{m}\sum_y \min\Big(\Delta, m_D(S,y) + f\left(S^*,y,V^1_{c_*}\right) - f\left(S,y,V^1_{c_*}\right)$$
$$+ f\left(S^*,y,V^1_{c_*}\cap V^2_{c(S^*,y)}\right) - f\left(S,y,V^1_{c_*}\cap V^2_{c(S,y)}\right)$$
$$+ |V^1_{C(S^*,y)}| - |V^1_{C(S,y)}|\Big).$$

The definition of $c(S,y)$ implies that

$$|V^1_{c(S^*,y)}| - |V^1_{c(S,y)} \leq f\left(S,y,V^1_{c_*}\cap V^2_{c(S,y)}\right) - f\left(S,y,V^1_{c_*}\cap V^2_{c(S^*,y)}\right)$$

so we have

$$\frac{1}{m}\sum_y \min\left(\Delta, m_D(S^*,y)\right)$$
$$\leq \frac{1}{m}\sum_y \min\left(\Delta, m_D(S,y) + f\left(S^*,y,V^1_{c_*}\right) + f\left(S^*,y,V^1_{c_*}\cap V^2_{c(S^*,y)}\right)\right)$$
$$\leq \frac{1}{m}\sum_y \min\left(\Delta, m_D(S,y) + 2f\left(S^*,y,V^1_{c_*}\right)\right)$$

and now the theorem follows from the same argument as in Theorem 5.3. □

2 Additional experimental results

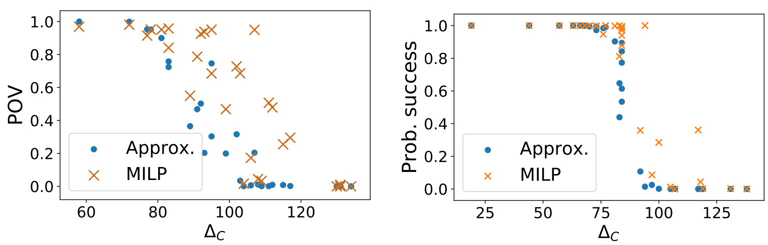

Figure 1: Probability of victory in constructive control. Left: irvine. Right: facebook

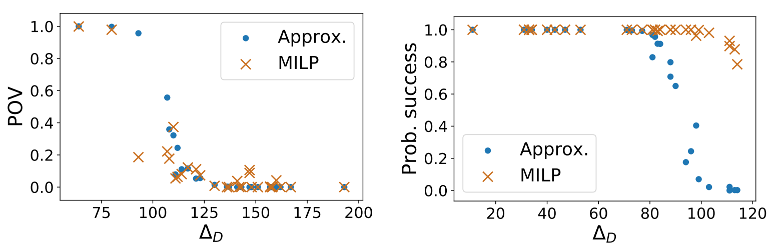

Figure 2: Probability of victory in destructive control. Left: irvine. Right: facebook. On irvine, the MILP was terminated after 24 hours, and had not found competitive solutions with the approximation algorithm on the intermediate margin instances by that time.