The Team Formation problem is NP-Hard even if we had a constant-time oracle that gives the value of each team.

Proof. Reduction to Knapsack from Team Formation.

Team Formation: Given a set of resources \( r_i \in R \), each with a cost \( b_i \in B \) we want to choose a set or resources \( S \subset R \) which maximizes the game value function \( F(S) \), and which has a total cost less than \( B \).

\[
\max_{S \in R} \left\{ F(S) : \sum_{i \in S} b_i \leq B \right\}
\]  

Knapsack: Given a set of \( n \) items with non-negative weights \( w_i \), and values \( v_i \) is there a subset of items with total weight at most \( W \), such that the corresponding value is at least \( V \)?

\[
\max \left\{ \sum_{i=1}^{n} v_i x_i : \sum_{i=1}^{n} w_i x_i \leq W \right\}
\]

We reduce the Knapsack problem to the team formation problem in the following way. We set the total number of resources equal to the number of items in the knapsack \( |R| = n \), with a budget equal to the knapsack capacity \( B = W \) and the cost of each resource equal to the cost of each item \( b_i = c_i \). Additionally each resource can cover a disjoint number of edges equal to the value of each item \( L_i = v_i \).

We then construct the game graph using a single source and a number of targets equal to the sum of all the profits \( |T| = \sum_i v_i \), with the value of each target is equal to the total number of targets \( r_i = |T| \). Because the paths to each of these targets are disjoint, maxi-min strategy for the attacker and defender will always be a uniform distribution over these paths. The game value of any game with \( p \) attacker paths will then be proportional to the number of edges \( E \) the defender can cover \( F(S) = -\sum_p x_i a_i - (1/p \times E/p) p^2 \). Because of the way the resource coverages were constructed, the total number of edges that can be covered by a team will be equal to the total value of all the items in the knapsack and the game value can be written as \( F(S) = \sum_i v_i r_i \).

The solution to the team formation problem will be equal to the maximum value of objects which can be placed in the knapsack since, both problems are then maximizing the same objective function, with the same constraints. The solution to the knapsack problem is therefore the solution to the team formation problem, and vice-versa.

Lemma 1 If the Compact Oracle strategy space contains the full strategy space of the DefenderOracle, then the game value of the Compact Problem will never be less than the true game value.

Proof. The strategy space of an oracle is defined by set of feasible assignments of values for their decision variables which determine the values which can be achieved for their utility function, where the best response strategy is the strategy which maximizes this function. Assume that the he set of feasible assignments in the Defender Oracle strategy space is smaller, and contained within the strategy space of the Compact Oracle. When the oracles are asked to compute a best response, one of two things can happen: Since the unrestricted strategy space always contains the restricted strategy space (1) either both oracles will return the best response if it exists in the restricted strategy space or (2) if the best response exists outside the restricted strategy space the restricted oracle will return a suboptimal strategy while the unrestricted oracle will return the best response. In the first case both oracles achieve the same utility, while in the second the restricted oracle will achieve a lower utility. The same logic can be applied to the Attacker Oracle playing with a full strategy space and when restricted to use only the min-cut paths.

We now compare the game values played in the Optimal Layer (Defender Oracle vs Attacker Oracle), the Reduced Layer (Defender Oracle vs Restricted Attacker Oracle) and the Compact Layer (Compact Oracle vs Restricted Attacker). By the previous statement, it is obvious that the utility of Reduced Layer, the Defender Oracle playing against the restricted Attacker Oracle will be greater than the Optimal Layer, if it were playing against the unrestricted attacker oracle. Similarly, the Compact Layer, will always achieve a better utility than the Reduced Layer. Since all oracles are still best response oracles (always selecting the best available strategy in their strategy space), even though they may play different strategies, the game played with the compact layer will always converge to a greater value than the optimal layer.

Theorem 2 The Defender Oracle’s strategy space is a subset of the Compact Defender Oracle strategy space.

Proof. We show any optimal strategy can be represented in the compact strategy space. The true number of resources of type \( k \) which can cover path \( A_j \) is the sum of all resources \( N_{S_i}^k \) assigned to cover any subset of paths \( S \) which include path \( A_j \), making sure not to double count. The compact representation only considers pairwise distances between paths, and approximates any terms where \( |S_i| > 2 \) as the number of pairs of paths \( \{a, b\} \) which can be reached from path \( j \).

\[
\sum_{D(S_i) \leq L_k} N_{S_i}^k \quad \text{s.t.} \quad j \in S \quad \rightarrow \quad N_j^k + \sum_{D(j,a),D(j,b),D(a,b) \leq L_k} N_{a,b}^k
\]  

LHS: Each \( N_{S_i}^k \) belongs to \( S_i \),the set of \( i \) attacker paths which can be covered by a resource of size \( L_k \). The sum then counts the size of the union \( \bigcup_{i=1}^{n} S_i \). RHS: Each pairwise assignment \( N_{a,b}^k \) belongs to \( S_{a,b} \), the set of pairwise attacker paths \( A_a \) and \( A_b \) which satisfy the condition \( D(j,a) \land D(j,b) \land D(a,b) \leq L_k \). The distance functions \( D \)
are constructed to use the minimum distance needed to cover a set of paths so that $D(S) \leq D(S')$ if $S \subseteq S'$. Therefore if $S'$ can be covered by a path of size $L_k$, so can $S$. This means that any pair of attacker paths in $\bigcup_{i=3}^{\infty} S_i$ must also be in the set $S_{a,b}$. Any assignment of resources on the LHS can be represented on the RHS, therefore any strategy (coverage of paths) which is feasible in the defender oracle strategy space must be feasible in the compact oracle’s strategy space. □